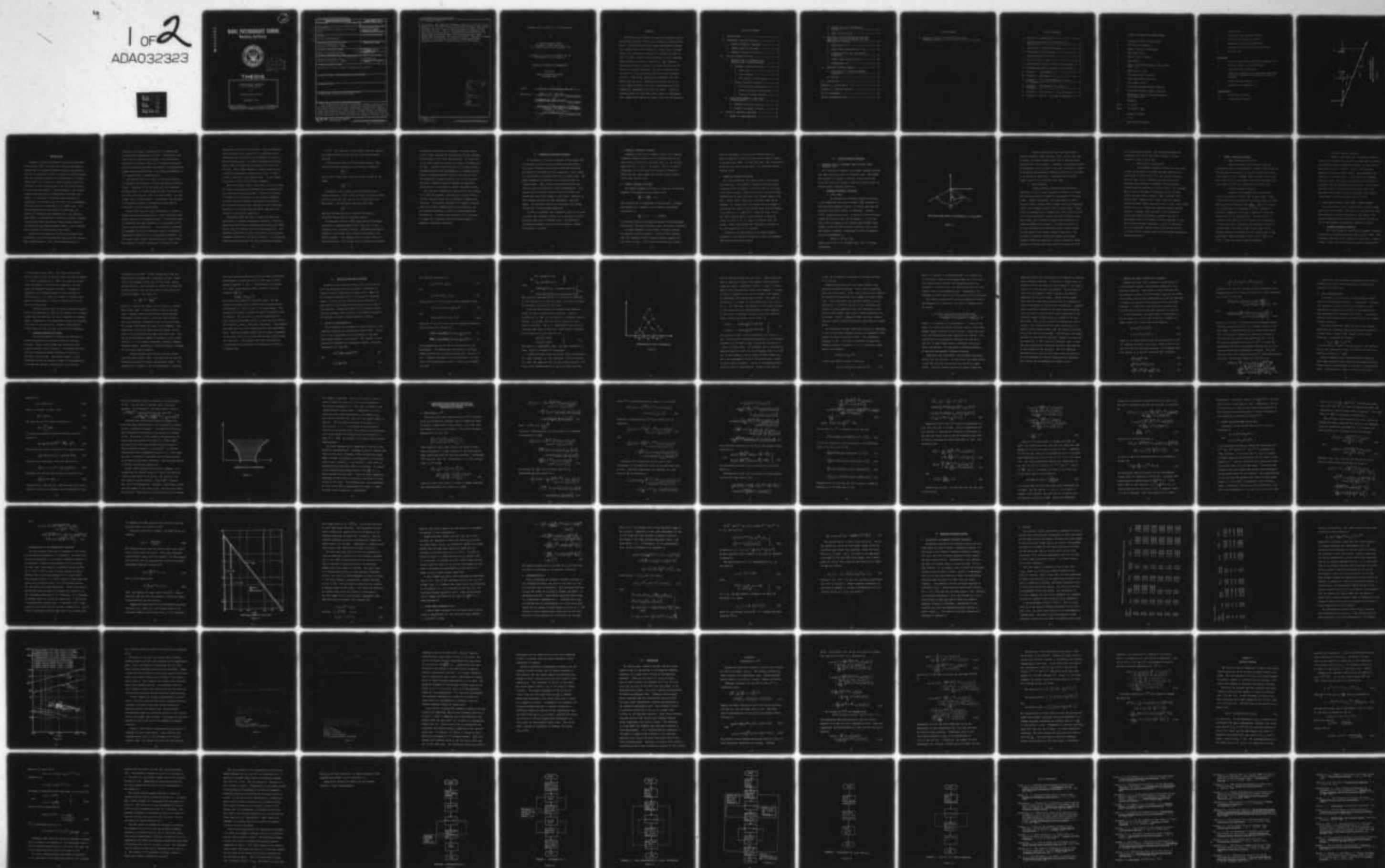
AD-A032 323 NAVAL POSTGRADUATE SCHOOL MONTEREY CALIF F/G 20/14
BREAKING WAVE CRITERION ON A SLOPING BEACH.(U)
SEP 76 R M SMITH

UNCLASSIFIED NL

1 OF 2
ADA032323



NAVAL POSTGRADUATE SCHOOL
Monterey, California

THESIS

BREAKING WAVE CRITERION
ON A SLOPING BEACH

by

Richard Markley Smith

September 1976

Thesis Advisor: E. B. Thornton

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Breaking Wave Criterion on a Sloping Beach. (6)		5. TYPE OF REPORT & PERIOD COVERED (9) Master's Thesis, September 1976
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) Richard Markley/Smith		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		12. REPORT DATE (11) September 1976
		13. NUMBER OF PAGES 97
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) Naval Postgraduate School Monterey, California 93940 (12) 95p.		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Breaking waves, wave theory, water waves

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The various wave theories, theoretical breaking criteria and derived breaking criteria are reviewed for shallow water waves. To account for the non-linear hydrodynamics present in a shallow water wave breaking on a beach with a sloping bottom, the perturbation technique of Iwagaki and Sakai is used to derive a second order expression for the horizontal water particle velocity for → next page

DD FORM 1473 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE
(Page 1) S/N 0102-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. Abstract (Cont'd)

eta

long waves. The kinematic breaking criterion is applied to the derived $c^{(2)}$ and $u^{(2)}$ values to establish breaking. The results indicate that the ratios of η_b/L_o and h_b/H_o provide reliable breaking criteria. Each of the parameters is dependent only upon beach slope and H_o/L_o. Theoretically derived values for h_b/H_o compare favorably with field measurements and offer significant improvement over previous theory. Predicted breaking depths are less than those present in experimental data, suggesting extension to higher orders may be warranted.

Breaking Wave Criterion on a Sloping Beach

by

Richard Markley Smith
Lieutenant, United States Navy
B.S., United States Naval Academy, 1971

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN OCEANOGRAPHY

from the

NAVAL POSTGRADUATE SCHOOL
September 1976

Author Richard Markley Smith

Approved by: Edward B. Thornton
Thesis Advisor

J. von Schwind
Second Reader

Warren C. Thompson, Acting
Chairman, Department of Oceanography

J. R. Borsting
Academic Dean

ABSTRACT

The various wave theories, theoretical breaking criteria and derived breaking criteria are reviewed for shallow water waves. To account for the non-linear hydrodynamics present in a shallow water wave breaking on a beach with a sloping bottom, the perturbation technique of Iwagaki and Sakai is used to derive a second order expression for the horizontal water particle velocity for long waves. The kinematic breaking criterion is applied to the derived $c^{(2)}$ and $u^{(2)}$ values to establish breaking. The results indicate that the ratios of η_b/L_o and h_b/H_o provide reliable breaking criteria. Each of the parameters is dependent only upon beach slope and H_o/L_o. Theoretically derived values for h_b/H_o compare favorably with field measurements and offer significant improvement over previous theory. Predicted breaking depths are less than those present in experimental data, suggesting extension to higher orders may be warranted.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF DRAWINGS

TABLE OF SYMBOLS AND ABBREVIATIONS

a	Constant Related to Wave Height
α	Perturbation Parameter
β	Angle of Bottom to Horizontal
c	Wave Phase Speed
g	Gravitational Constant
H	Wave Height
h	Depth of Water Referenced to Still Water Level = $i \cdot x$
i	Beach Slope = $\tan \beta$
J_o	Zero-Order Bessel Function
J_1	First-Order Bessel Function
L	Wave Length (feet)
N_o	Zero-Order Neumann (Weber) Function
N_1	First-Order Neumann (Weber) Function
η	Elevation of Free Surface Referenced to Still Water Level
p	Pressure
p(x)	$2\sigma(x/gi)^{\frac{1}{2}}$
$\varphi(x)$	$2\sigma(x/gi)^{\frac{1}{2}} - \frac{\pi}{4}$
ρ	Density of Water
σ	$2\pi/T$
T	Wave Period (seconds)

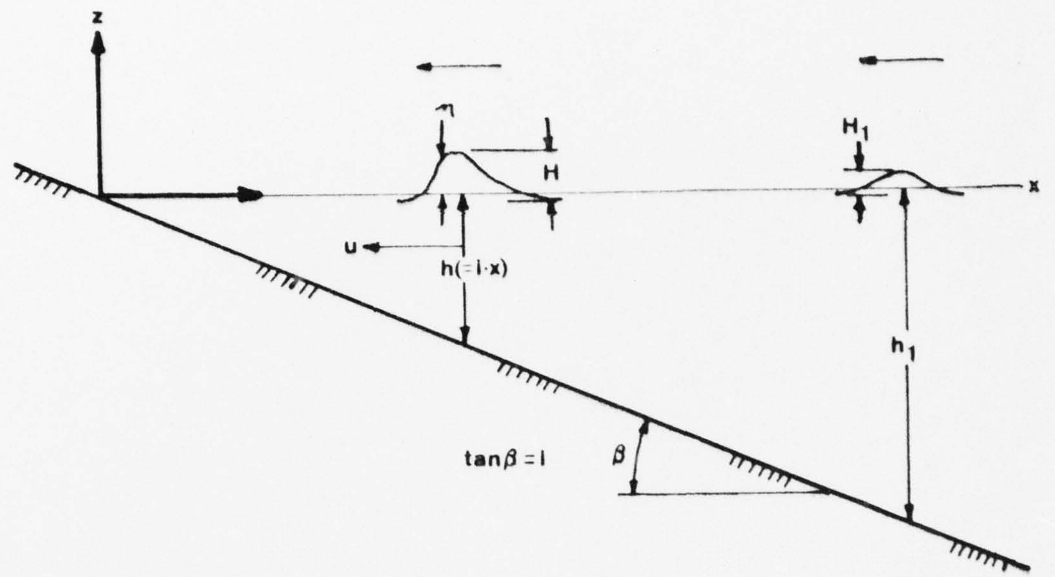
t	Time (seconds)
u	Horizontal Water Particle Velocity
w	Vertical Water Particle Velocity
x	Horizontal Distance from the Beach
y	Elevation of Wave Crest Above the Bottom
z	Vertical Axis

Subscripts

b	Value of Quantity at Point Where Breaking Occurs
c	Value of Quantity at the Wave Crest
o	Deep Water Wave Conditions
1	Conditions present at the point where the depth of water is maximized in applicable range of solution.
x	Derivative with Respect to x
t	Derivative with Respect to t

Superscripts

(1)	First Order Solution
(2)	Second Order Solution



SYMBOL DEFINITIONS:
WAVES ON A SLOPING BEACH

Figure 1

I. INTRODUCTION

Attempts at deriving breaking criteria have been made since Stokes (1847) presented his classical development. Derivations of the many available water wave theories all involve the solution of Euler's equations of motion coupled with the continuity equation for incompressible, inviscid, irrotational flow subject to certain boundary conditions. Breaking, or near breaking waves have very steep profiles in which the wave height is large compared to the relevant length scale implying the hydrodynamics are highly non-linear. At the onset of breaking strong vorticity is introduced at the surface near the crest and the assumption that the motion is irrotational is no longer valid. The strong non-linearities and induced vorticity make the analysis of breaking waves mathematically very difficult. This thesis is concerned with finding an incipient breaking criterion for waves shoaling on a beach before vorticity is induced but including non-linear effects. The discussion will be limited to shallow water wave theory.

Solution of the equations of motion has required the application of physical assumptions associated with various wave characteristics. Thus, each formulated theory is limited in its range of applicability to regions where its underlying assumptions are valid. Shallow water wave theory may be classified according to the bottom being horizontal or sloping. This is an important restriction because field and laboratory measurements of breaking waves suggest that the bottom slope is an important parameter in the classification of breaking waves.

The simplest form of solution to the wave problem is to linearize the equations of motion assuming a horizontal bottom. Peregrine (1972) has shown that the linearized equations apply to regions where the ratio H/L and H/h are both much less than one, where H is wave height, L is wave length, and h is water depth. Consequently, this solution is restricted to waves of infintesimal height and not applicable to steep breaking waves.

Stokes (1847) was the first investigator to present a higher order solution applicable to finite amplitude waves, though limited to a horizontal bottom. In his development, he transforms the basic equations to an equivalent set by using a velocity potential, ϕ . The solution is obtained by expanding the velocity potential using a perturbation scheme which employs H/L as a perturbation parameter. To the lowest order, Stokes' method results in linear theory. The accuracy of Stokes' solution at a particular order decreases as the ratio h/L decreases. Dean and Eagleson (1966) attribute this inaccuracy to increasing bottom influences and a decrease in the importance of vertical particle acceleration. De (1955) concluded that Stokian theory should be discarded for values of h/L of 0.125 and less. Dean (1968) expanded a stream function using a numerical perturbation scheme and was able to raise a "Stokes" type wave to any desired order. In this manner he was able to solve for incipient breaking.

When the relative depth is very small, as in very shallow water, the vertical acceleration can be neglected and the fluid path curvature is small. Hence, the pressure is assumed to be hydrostatic as the vertical component of motion does not influence the pressure distribution. The resulting equations are referred to as the "long wave equations." A sloping bottom and finite amplitude are allowed by the long wave equations.

Freidrichs (1948) was able to derive the long wave equations by a rigorous mathematical approach. Utilizing quantities h and L which represent typical depth and length scales, the Airy equations were non-dimensionalized. This procedure resulted in a large stretching of the horizontal coordinate relative to the depth coordinate. A perturbation analysis was then applied with the perturbation parameter

$\sigma = h^2/L^2$. He found that to the lowest order the pressure was indeed hydrostatic and that the long wave equations resulted.

A third length scale for shallow water theory, which utilizes the wave amplitude a, is the Ursell parameter

$$\frac{a}{L}\frac{L}{h}^{3} .$$

Ursell (1953) showed that long wave theory belongs to the regime

$$\frac{a}{L}\frac{L}{h}^{3} \gg 1$$

Boussinesq (1872) assumed that the pressure was no longer hydrostatic, which allows inclusion of vertical water particle velocities, but results in a limitation on the wave height. The Boussinesq equations apply when

$$\frac{a}{h}\frac{L}{h}^{3} \sim 1$$

implying the waves are not as high and the water is relatively deeper than for long wave theory.

Korteweg and de Vries (1895) simplified the Boussinesq equation by considering waves which travel only in one direction over a horizontal bottom. Extending Boussinesq's equation in this manner they produced a wave theory they termed "cnoidal." The limiting case of cnoidal theory is the solitary wave. The cnoidal/solitary theory has received considerable attention by researchers in recent years. Keller (1948) extended the perturbation analysis employed by Freidrichs to the first approximation. He found that to the first order his results were those of Korteweg and de Vries. Laitone (1960) continued the process further, obtaining second order approximations to cnoidal/solitary waves by solving Freidrichs' method to the fourth order. The assumption of waves traveling only in one direction precludes a reflected wave and imposes the important restriction of a horizontal or nearly horizontal bottom.

The long wave equations are used in this study because it is felt that properly including the sloping bottom is the most important next step in seeking a breaking wave criterion. The possible importance of vertical accelerations in the wave breaking process are recognized, but are assumed negligible in order to obtain mathematical tractability. A second order solution of the long waves is sought and a breaking criterion derived based on a kinematic instability condition.

II. THEORETICAL BREAKING CRITERIA

It is desired to formulate breaking criteria which can be expressed in terms easily observable and measurable. The several breaking criteria which have been developed may generally be broken into two categories. First, there are those which are derived from waves of steady form. The waves considered are assumed to be in shallow water of constant depth. Both Stokian and cnoidal/solitary wave theories have been employed in these investigations. The second group of derived criteria consists of those concerned with waves which deform as they shoal. Research in this category of waves has been confined to long wave theory. The derived criteria for horizontal and sloping bottoms will be investigated separately.

In order to determine wave parameters which can be used to predict the breaking of waves, it is necessary to first formulate some type of limiting value. The physically significant breaking criteria are the kinematic, dynamic and geometric criteria.

A. KINEMATIC BREAKING CRITERION

Originally formulated by Rankine (1864), the kinematic breaking criterion states that the limiting value of the water particle velocity at the wave crest, u_c, is the wave phase velocity, c, $u_c \leq c$. Physically, this is a logical limitation, for if the particle velocity is allowed to exceed the wave phase speed, the particle would separate itself from the wave form.

B. DYNAMIC BREAKING CRITERION

The dynamic breaking criterion is stated by the vertical momentum flux equation at the surface ($z=\eta$)

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g .$$

The criterion can be formulated in several ways. Assuming the pressure is a constant at the surface, the maximum acceleration is

$$\frac{Dw}{Dt} \leq g \qquad ; \rho \text{ constant}$$

If the water particle acceleration exceeds the gravitational acceleration, the water particles leave the surface vertically.

A second statement of the dynamic criterion concerns the vertical pressure gradient. Laitone (1963) proposed that the limiting vertical pressure gradient beneath the wave crest is zero. In his study of cnoidal and solitary waves he determined a value for H/h beyond which the pressure gradient reverses its sign and the pressure begins to decrease with depth. He concludes that this is physically impossible and thus accepts zero as the limiting pressure gradient value.

C. GEOMETRIC BREAKING CRITERION

As a wave progresses into shallow water, the surface slope steepens. The geometric breaking criterion places a limiting value of infinity (vertical face) on the slope of the water surface. Beyond this value, the wave becomes unstable and the water particles fall forward ahead of the wave. Stoker (1957) shows that a vertical slope can be obtained. An insight into the concept can be obtained by considering the speed of the shallow water wave disturbance to be given by $c = [g(\eta + h)]^{\frac{1}{2}}$. Since the crest of the wave has a greater depth of water beneath it than the trough in front of it, it tends to "catch-up" with the trough. Hence, the forward face continues to steepen as the wave shoals until it is vertical.

A review of the application of the three breaking criterion to the separate categories of waves of permanent form and shoaling waves follows.

III. DERIVED BREAKING CRITERIA

A. BREAKING WAVES OF PERMANENT FORM (SHALLOW WATER, CONSTANT DEPTH)

The theoretical kinematic and dynamic breaking criteria have been applied to waves of permanent form. The former has resulted in a number of derived criteria while the latter has only been applied to cnoidal/solitary theory to produce single breaking parameters.

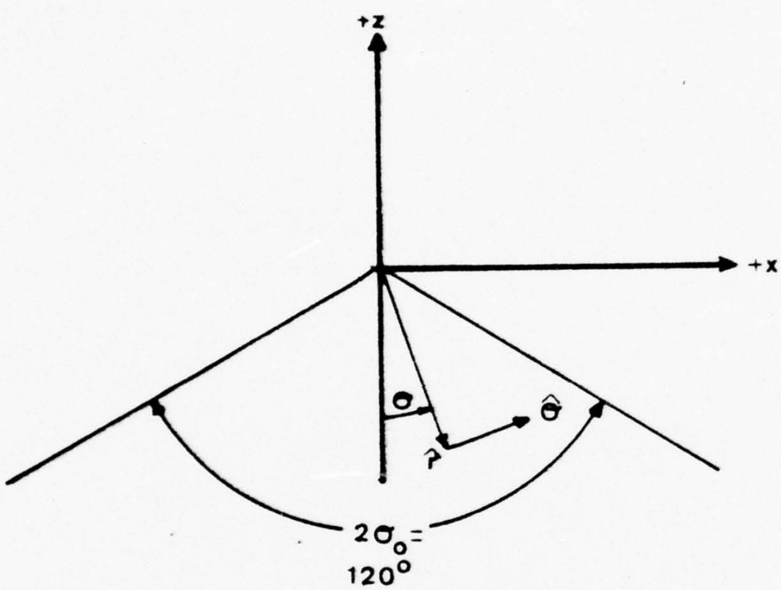
1. Kinematic Breaking Criterion

a. Crest Angle

By applying the kinematic breaking criterion to his formulated wave theory, Stokes (1880) produced the first derived breaking criteria. He showed that when the enclosed crest angle, Figure 2, reached $\frac{2}{3}\pi$ radians, (120°), breaking would occur. To arrive at this conclusion Stokes was forced to make two assumptions: 1) that the crest would be formed by two intersecting straight lines tangent to the real water surface curvature, and 2) that the velocity potential, transformed to polar coordinates, could be approximated by

$$\phi(r,\theta) = Br^{N} \sin (N\theta),$$

where B and N are to be evaluated and r and θ are polar coordinates.



ENCLOSED CREST ANGLE FOR KINEMATICALLY LIMITED WAVE

Figure 2

Several investigators have verified Stokes' criteria (Gaughan, Komar and Nath, 1973), however the range of depths over which Stokian theory can be applied places limitations on this criteria. Therefore, a contribution made by Chappelear (1959) in which he was able to develop a method of satisfying the kinematic breaking criterion for all depths is of particular importance. Chappelear also verified Stokes' value for the limiting crest angle.

b. Wave Steepness

Another breaking criteria which has gained acceptance is that of wave steepness in which it is stated that the wave height is limited to one-seventh of the wave length. Several researchers, including Michell (1893), Havelock (1918), Davies (1952), Yamada (1957) and Chappelear (1959), have produced results close to this figure. Their values for maximum wave steepness vary from 0.1412 to 0.1443. All of their derivations closely followed classical lines. Dean (1968) used a numerical stream function approach to examine breaking wave criteria. He found a solution to the full nonlinear wave formulation that is exact except for the dynamic free surface boundary conditions. The solution is found by fitting the dynamic free surface boundary conditions numerically using an iterative scheme. Taking the limiting horizontal particle velocity as 98.5%

of the wave phase velocity, the resulting maximum wave steepness value was 0.1723, which differs from those previously given by about 20%.

c. Wave Height to Depth Ratio

The kinematic breaking criterion has been used by many investigators to obtain limiting values for the wave height to depth ratio, H/h. The results, however, have not been at all consistent and vary widely with different wave theories. Chappelear (1959) analyzed Stokian waves and arrived at a value of 0.87. For solitary waves, results obtained by Boussinesq (1871), Rayleigh (1876), McCowan (1894), Gwyther (1900), Packhan (1952), Davies (1952), Yamada (1957), Lenau (1966) and Yamada, Kimura and Okabe (1968), vary from 0.73 to 1.03. Dean (1968) found a value of 1.0 using his numerical stream function approach. Gaughan, Komar and Nath (1973) express the opinion that these differences probably arise due to approximate fits of the complex velocity potential to the free surface boundary conditions. As will be discussed in a later section, Laitone (1963) found values of H/h = 0.73 and 0.81 for solitary waves using different theoretical breaking criterion.

2. Dynamic Breaking Criterion

a. Water Particle Acceleration

Various limiting values of water particle acceleration have been determined. Kinsman (1965) gives the gravitational acceleration, g, as the limiting value for a crest angle of 120°. Gaughan, Komar and Nath (1973), however, show that by using Stokes' wave crest equations and a crest angle of 120°, the limiting value should be $\frac{g}{2}$.

b. Vertical Particle Velocity

Laitone (1963) examined the vertical water particle velocity for cnoidal and solitary waves. Using a third order velocity equation for cnoidal waves, he found that values of H/h greater than $\frac{8\beta}{9\beta+2}$, where β, a property of the wave, is restricted to $\frac{1}{2}<\beta\leq 1$, produced physically impossible velocities. Hence this ratio was established as the limiting value. As the value of β approaches the limit of 1.0 the cnoidal wave approaches a solitary wave form. Therefore, the limiting value for a solitary wave to the third order is $H/h = \frac{8}{11} = 0.7272$. Laitone carried his solution for the solitary wave to the next higher order and found a value of $H/h = \sqrt{3} - 1 = 0.7321$. These two results compare favorably.

c. Vertical Pressure Gradient

Laitone (1963) developed a different limiting value for H/h than that previously discussed when he applied the vertical pressure gradient criterion. In this case he found that for $H/h = (2\beta/3)^{\frac{1}{2}}$, $\frac{1}{2} < \beta \leq 1$, the pressure gradient is zero for cnoidal waves. This expression was derived to the third order. If the ratio H/h increased from this value, the sign of the gradient would reverse, a condition that he concluded could not exist. In the limiting case of $\beta = 1.0$, which gives a solitary wave, the limiting value of H/h = 0.812 is obtained. This differs significantly from his previous limit of 0.7272.

B. WAVES WHICH DEFORM AS THEY SHOAL (SLOPING BOTTOM)

As a wave shoals over a sloping bottom, the wave height and profile are altered. The theory of long waves has been most generally applied to research in this region. The derived breaking criteria have been formulated through the use of the kinematic and geometric breaking criteria.

1. Kinematic Breaking Criterion

The first attempt at applying the kinematic breaking criterion to a deforming wave was made by Ayyar (1970). His derivation made use of the concept of a wave front. Simply stated, a wave front is the position where a discontinuity in the surface slope occurs. The slope of the surface will be zero in front of the wave front and negative behind it. Ayyar's approach was to obtain the slope at the wave front, integrate to find the free surface η , and then apply the kinematic breaking criterion. Assuming the geometry of the plunging breaker, he then derived the value of $y_b/h_b = 2.0$, where y_b = height of breaker crest above the bottom and h_b = depth at breaking point below the still water level.

Several problems exist in Ayyar's derived criterion. First, the derivation is based on the geometry of a plunging breaker and thus excludes the other categories of breaking waves. Additionally, his formulation assumes that breaking will occur at the wave front. Gaughan, Komar and Nath (1973) point out that this may not be a valid assumption.

2. <u>Geometric Breaking Criterion</u>

Use of the geometric criterion has been made by several researchers in developing long wave breaking criteria. Stoker (1957) showed that long waves could obtain a surface slope of infinity. He extended his work and used a numerical methods technique to arrive at a solution to the problem. The method, however, is not satisfying in that it requires a number of approximations to be made and requires recalculation as the initial conditions are altered. Further discussion of this procedure will be presented in a subsequent section. Burger (1967) and Greenspan (1958) used the wave front concept and the vertical slope criterion to predict the horizontal distance traveled from the wave front at time t=0 to the point of breaking. The result was

$$x_b = \frac{h_1}{M} \left(1 - \frac{2S}{S+M}\right)^{4/3}$$

where, M = slope of the beach, $h(x) = h_1 - Mx$, h_1 = initial water depth, and S = initial surface slope at the wave front. Gaughan, Komar and Nath (1973) discuss several limitations to this result. As was the case in Ayyar's work, the breaking is assumed to occur at the wave front. The surface slope behind the front is not examined. Some other point, such as the wave crest, may become vertical prior to this condition occurring at the wave front. Also, the use of horizontal distance to breaking is not a useful criteria. It is not easily measurable, having a somewhat arbitrary origin. Prediction of the wave height at breaking is a much more useful parameter.

Another approach involving the vertical surface slope was taken by Mei (1966). The basis for his work was originally proposed by Carrier and Greenspan (1959). The technique is to produce a set of characteristic equations from the long wave equations and then to make an additional transformation through the use of a final pair of independent variables, σ and λ . The equations are reduced to a single linear equation which involves a velocity potential, $\Phi(\sigma, \lambda)$,

$$(\sigma\Phi)_\sigma - \sigma\Phi_{\lambda\lambda} = 0$$

Mei solves this equation to the first order. He then follows a procedure used by Carrier and Greenspan whereby the Jacobian $J = \partial(x,t)/\partial(\sigma, \lambda)$ is investigated. This Jacobian will vanish at points for which the surface slope is infinite. Mei was able to obtain an expression for h_b/H_o which was dependent upon H_o, L_o and the bottom slope. The subscript $_o$ denotes deep water conditions. Unfortunately, Mei found that his theory compared poorly with experimental data. Predicted breaking depths were too large, dependence on the beach slope was too great and the wave profiles were too sinusoidal. Mei suggests that these discrepancies could possibly be eliminated by extending the solution to a higher order.

IV. REVIEW OF LONG WAVE SOLUTIONS

Essentially two approaches exist which can be used to solve the nonlinear long wave equations. The first procedure, initially formulated by Stoker (1947), makes use of a solution technique known as the method of characteristics. A final solution utilizing this method can be made either through numerical calculation or by an analytical approach. Iwagaki and Sakai (1972) propose a second solution procedure which involves a perturbation expansion. An evaluation of each of these techniques follows.

A. METHOD OF CHARACTERISTICS

The application of the method of characteristics to the long wave equations was explained by Stoker (1958). Necessary to this development is the acceptance of the wave phase speed relation $c = [g(\eta + h)]^{\frac{1}{2}}$. The validity of this equation is discussed in a subsequent section. From this expression it is seen that

$$c_x = (g\eta_x + gh_x)/2c \qquad (1)$$

and

$$c_t = g\eta_t/2c. \qquad (2)$$

The long wave equations are

$$u_t + u u_x = -g\eta_x \tag{3}$$

and

$$[u(\eta + h)]_x = -\eta_t \ . \tag{4}$$

Substitution of (1) and (2) into these equations yields

$$u_t + u u_x + 2cc_x - g h_x = 0 \tag{3a}$$

and

$$2c_t + 2uc_x + cu_x = 0 \tag{4a}$$

Relations (5) and (6) result from the respective addition and subtraction of (3a) and (4a),

$$\{\partial/\partial t + (u+c)\partial/\partial x\}\cdot(u + 2c - gh_x t) = 0 \tag{5}$$

$$\{\partial/\partial t + (u-c)\partial/\partial x\}\cdot(u - 2c - gh_x t) = 0 . \tag{6}$$

The interpretation of (5) and (6) is essential to the development. (5) implies that the function $(u+2c-gh_x t)$ remains constant for a particle moving with a velocity of u+c. A similar evaluation of (6) can be made. In other words, two characteristic curves, C_1 and C_2, are defined such that

$$\left.\begin{aligned} C_1 &: dx/dt = u + c \\ \text{and} \qquad & \\ C_2 &: dx/dt = u - c \end{aligned}\right\} \tag{7}$$

where

$$\left.\begin{aligned} u + 2c - gh_x t &= K_1 = \text{A CONSTANT ALONG } C_1 \\ \text{and} \qquad & \\ u - 2c - gh_x t &= K_2 = \text{A CONSTANT ALONG } C_2 \end{aligned}\right\} \tag{8}$$

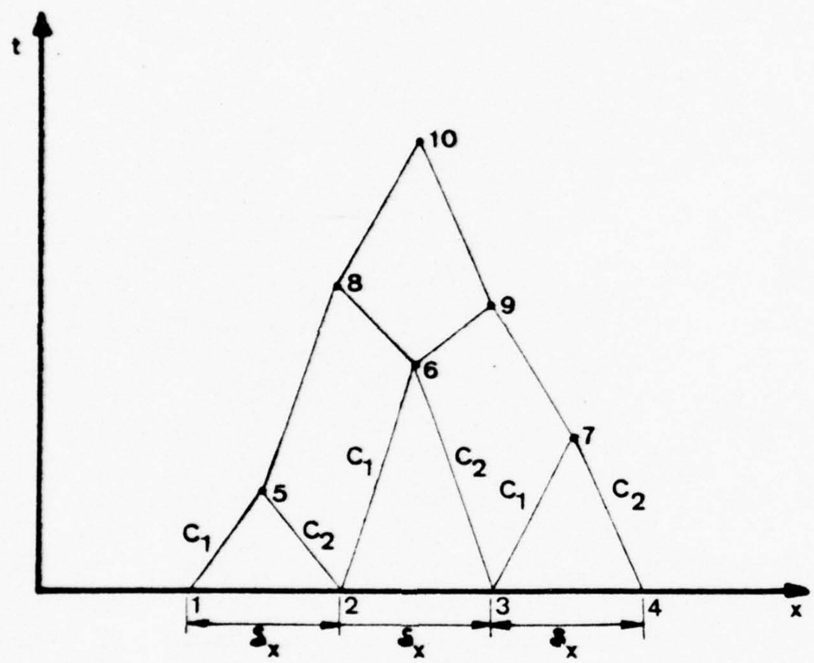
The system of equations given by (7) and (8) is equivalent to that defined by (3a) and (4a), hence, a solution of either set provides a solution for the other.

It now becomes necessary to determine the characteristics so that a solution may be obtained. Stoker's technique was to make use of a method of successive approximations. He assumed that the initial values of u and η could be observed. Use of $c = [g(\eta + h)]^{\frac{1}{2}}$ could then provide the phase speed at time t=0. These initial conditions are assigned values such that for t=0

$$\left.\begin{aligned} u(x,0) &= \bar{u}(x) \\ c(x,0) &= \bar{c}(x) \end{aligned}\right\} \tag{9}$$

The task is to approximate u and c for small increments of time. Figure 3 clarifies the discussion.

A series of points along the x-axis, which are separated by a small distance δ_x, are considered. Since the values of u and c are known for each of these points from (9), the slope of the characteristics C_1 and C_2 at each point can



INTEGRATION BY FINITE DIFFERENCES

Figure 3

also be obtained through the use of (7). These slopes are used to construct straight line segments from the points along the x-axis. Location of points 5, 6 and 7 is determined by the intersection of these line segments. A source of error is inherent in the use of straight line segments to approximate the characteristic curves. This error is restricted to a minimum by using sufficiently small values of δ_x. For this case, the tangents to the curves provided by the slopes give good approximations to small segments of the curves. Equations (8) and (9) give the characteristics issuing from points on the x-axis as

$$\left.\begin{aligned} &\text{along } C_1: \quad u + 2c - gh_x t = \bar{u} + 2\bar{c} \\ &\text{along } C_2: \quad u - 2c - gh_x t = \bar{u} - 2\bar{c} \end{aligned}\right\} \qquad (10)$$

The values of x and t can be obtained for the points 5, 6 and 7 (this could be accomplished graphically, for instance). Equation (10) can then be used to determine the values of u and c at these points. The procedure can be continued to obtain values for u and c at the subsequent points 8, 9 and 10. In this manner, a net of points at which values of u and c are approximated could be constructed which covered an entire field of concern. Values at intermediate points could be found by interpolation. Stoker states that as

$\delta_x \rightarrow 0$, the process will converge to an unique solution of (7) and (8).

Numerous objections have been raised against this particular numerical approach. Ayyar (1970) categorizes these into three areas. First, the solution is not explicit and requires re-calculation for a change in initial conditions. This alone makes the procedure difficult to use effectively. A second criticism is that a spilling type breaker is always predicted; thus the elimination of the several other breaker types places a severe restriction on the method. Finally, LeMéhauté (1968) has given evidence that the technique produces incorrect predictions of breaker points.

An alternative and more analytical solution to equations (7) and (8) has been offered by Carrier and Greenspan (1958) and was discussed in Section III.B.2. Two independent variables σ and λ are used to transform the equations. Defining a velocity potential $\phi(\sigma, \lambda)$ yields the linear second order equation

$$(\sigma\phi_\sigma)_\sigma - \sigma\phi_{\lambda\lambda} = 0 \,, \tag{11}$$

to which the authors propose a solution,

$$\phi(\sigma,\lambda) = A\, J_0(\omega\sigma)\cos(\omega\lambda - \psi) \tag{12}$$

where A = constant to be determined and ψ = a phase lag. At this point, Carrier and Greenspan made use of the vertical surface slope criterion and determined values of a Jacobian, $J = \partial(x,t)/\partial(\sigma,\lambda)$, which would exist for the specific case of non-breaking waves. Consequently, the remainder of their study is of little value to our discussion.

Mei (1966) considered the solution technique of Carrier and Greenspan as it applied to the case of breaking waves. He selects

$$\psi = -B\left[J_0\left(\sigma/2i^{1/2}\right)\cos\left(\lambda/2i^{1/2}+\pi/4\right) + N_0\left(\sigma/2i^{1/2}\right)\sin\left(\lambda/2i^{1/2}+\pi/4\right)\right] \quad (13)$$

where B = a constant to be determined, i = slope of the beach, J_o = zero-order Bessel function, and N_o = zero-order Neumann (Weber) function as his solution. Using this relation, Mei derived first order equations for u and η. The coefficient B is determined by matching the solution to that for an outer region where a horizontal bottom Airy Theory was applied. Mei determined a breaking parameter by applying the geometric breaking criterion.

Comparison with experimental data indicated that Mei's results were not accurate. He attributed these deficiencies to the fact that the solution was not carried to higher orders. Previous studies conducted by 3enney (1966) had suggested higher order derivations were required for shallow water breaking conditions. Solution to a higher order is required in both the inner and outer regions if the coefficient B is to be accurately determined. The procedure of Tlapa, Mei and Eagleson (1966) provides a perturbation expansion for the outer region. Review of this method shows that the third order solution must be considered to uniquely determine the second order coefficients. Assuming that a successful solution to the outer region expansion could be found, or that B can be found independent of the off-shore regime, difficulties still remain in the near shore area. This is the location of breaking and thus of concern. The solution would be vastly simplified if a method could be found in which only the velocity potential used by Mei need be perturbated. Unfortunately, the single equation for the velocity potential prevents this approach. The quantities σ and λ are only variables used to transform the characteristic equations and hence perturbations on them produce no new equations. The only remaining alternative is to use the next higher order expression of the Airy equations to form the characteristic equations. This implies the consideration of cnoidal theory.

B. IWAGAKI AND SAKAI PERTURBATION TECHNIQUE

Iwagaki and Sakai (1972) proposed a second method of solution which involved a perturbation expansion of the long wave equations. The premise of their study was that the asymmetric profile of shallow water waves and the effect of the beach slope on wave transformations could be explained by taking into account the nonlinearity of the long wave equations. To show this, they developed a second order solution for η from the long wave equations (3) and (4).

In the derivation, the beach slope is represented by i and the depth, h, is such that h=h(x)=i·x. Iwagaki and Sakai assume that u and η may be expressed as

$$\eta = \alpha \eta^{(1)} + \alpha^2 \eta^{(2)} + \cdots \tag{14}$$

$$u = \alpha u^{(1)} + \alpha^2 u^{(2)} + \cdots \tag{15}$$

where α is a small quantity and the superscripts (1) and (2) indicate the order of the term. Substitution of (14) and (15) into the long wave equations, and arranging them with respect to α and α^2 yields the four equations,

$$u_t^{(1)} + g\eta_x^{(1)} = 0 \tag{16}$$

$$\eta_t^{(1)} + [u^{(1)}h]_x = 0 \tag{17}$$

$$u_t^{(2)} + u^{(1)}u_x^{(1)} + g\eta_x^{(2)} = 0 \tag{18}$$

$$\eta_t^{(2)} + \left[u^{(1)} \eta^{(1)} + u^{(2)} h \right]_x = 0 \tag{19}$$

Equations (16) and (17) together give two equations in two unknowns which can be solved to obtain first order expression for η and u of the form

$$\eta^{(1)}(x,t) = a \left[\cos\sigma t \cdot J_0\left(2\sigma\sqrt{x/gi}\right) - \sin\sigma t \cdot N_0\left(2\sigma\sqrt{x/gi}\right)\right] \tag{20}$$

and

$$u^{(1)}(x,t) = a\sqrt{g/i}\; x^{-\frac{1}{2}}\left[\sin\sigma t \cdot J_1\left(2\sigma\sqrt{x/gi}\right) + \cos\sigma t \cdot N_1\left(2\sigma\sqrt{x/gi}\right)\right] \tag{21}$$

where $J_r(\ )$ and $N_r(\ )$ represent Bessel and Neumann (Weber) functions respectively and a is a constant related to the wave height. Equations (20) and (21) can be substituted into (18) and (19), reducing the problem to solution of (18) and (19) for second order expressions for u and η. The authors presented a solution for $\eta^{(2)}$ in which they approximated the Bessel and Neumann functions in the first order solutions with trigonometric functions. They show that for values of x greater than $gi/4\sigma^2$ these substitutions are valid. The derived formula for $\eta^{(2)}(x,t)$ is

$$\eta^{(2)}(x,t) = \frac{a^2}{\pi i}\, x^{-1} \cos\left\{2\left(\sigma t + 2\sigma\left(\frac{x}{gi}\right)^{1/2} - \frac{\pi}{4}\right) + \frac{\pi}{2} + \tan^{-1}\left(\frac{3}{10}\frac{(gi)^{1/2}}{\sigma} x^{-1/2}\right)\right\} + \left[\frac{a^2}{4\pi\sigma}\left(\frac{g}{i}\right)^{1/2} x^{-3/2} \cos\left\{2\left(2\sigma\left(\frac{x}{gi}\right)^{1/2} - \frac{\pi}{4}\right)\right\} - \frac{a^2 g}{16\pi\sigma^2} x^{-2} \sin\left\{2\left(2\sigma\left(\frac{x}{gi}\right)^{1/2} - \frac{\pi}{4}\right)\right\}\right]. \tag{22}$$

Iwagaki and Sakai presented experimental results which confirmed that their solution technique produced valid results for the wave form.

C. PHASE SPEED RELATION

In several of the previous sections mention has been made of the phase speed relation, $c = [g(\eta + h)]^{\frac{1}{2}}$. Although this equation is widely used in the literature on wave theories and is generally accepted; few discussions have been presented which establish its validity. The question deserves some attention prior to application of the kinematic breaking criterion.

The first inclination toward the use of the relation comes from the linearized long wave equations for water of constant depth. Stoker (1958) showed that for these conditions u satisfies the relation

$$u_{xx} - \frac{1}{gh} u_{tt} = 0 .$$

η can be shown to satisfy the same equation. The implication of this expression is that the speed of the wave disturbance is given by $c = (gh)^{\frac{1}{2}}$.

Another indication that the expression may indeed be valid comes from what Stoker terms the gas dynamics analogy. Stoker credits the development of this idea to Riabouchinsky (1932). Consideration is made of a mass per unit area expressed by

$$\bar{\varrho} = \varrho(\eta + h) \tag{23}$$

where ϱ = density of water. Thus

$$\bar{\varrho}_t = \varrho \eta_t \, . \tag{24}$$

The force $\bar{p}$ per unit width is defined as

$$\bar{p} = \int_{-h}^{\eta} p \, dz \; . \tag{25}$$

By using the hydrostatic pressure relation, $\bar{p}$ may be reduced to

$$\bar{p} = g\varrho/2 \, (\eta + h)^2 = \frac{g}{2\varrho} \, \bar{\varrho}^2 . \tag{26}$$

Multiplying both sides of equation (3) by $\varrho(\eta + h)$ gives

$$\varrho(\eta + h)(u_t + u u_x) = -g \varrho (\eta + h) \eta_x \tag{27}$$

which may be re-expressed, using (23) and (26), as

$$\bar{\varrho}(u_t + u u_x) = -\bar{p}_x + g \bar{\varrho} h_x \, . \tag{28}$$

Equations (23) and (24) can be used to re-write (4) as

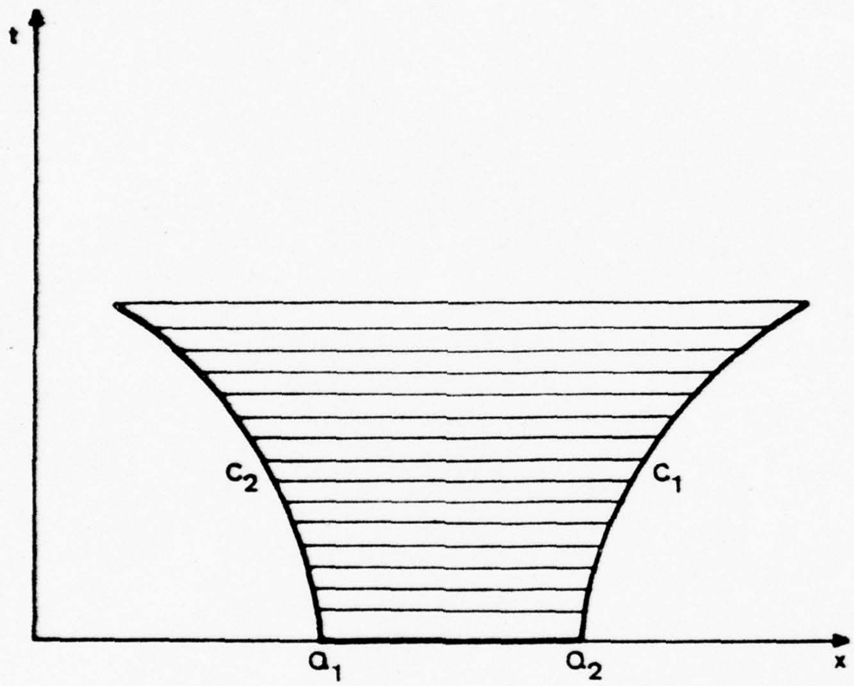
$$(\bar{\varrho} u)_x = -\bar{\varrho}_t \, . \tag{29}$$

Equations (26), (28) and (29), when combined, give results similar to those of gas dynamics for one dimensional flow, the only difference being the presence of the term $g\bar{\varrho}h_x$ in (28). For the case of constant depth, this term vanishes. In gas dynamics, the sound speed is given by $c = [d\bar{p}/d\bar{\varrho}]^{1/2}$. Applying this with (23) and (26),

$$c = [d\bar{p}/d\bar{\varrho}]^{1/2} = \left(\frac{2g\bar{\varrho}}{2\varrho}\right)^{1/2} = [g(\eta+h)]^{1/2}.$$

Although these discussions provide an insight into using the phase speed relation, they can hardly be considered as a definitive argument. A satisfactory derivation can be obtained by returning to the methods of characteristics. The review of this technique explained that the phase speed was assumed to be given by $c = [g(\eta+h)]^{\frac{1}{2}}$. The theory could have been derived equally as easily by simply defining a quantity $c = [g(\eta+h)]^{\frac{1}{2}}$. No physical significance need be immediately applied to c. This being the case, an identical development can be made resulting in the same characteristic equations. The task then becomes to discover the physical meaning of c.

Stoker (1958) presents the following argument. It is assumed that the initial values of u and η are given for a body of water which is in motion. The value of c for this instant is given through $c = [g(\eta+h)]^{\frac{1}{2}}$. Figure 4 will aid in the explanation. Consider a disturbance created over the segment of the x-axis Q_1 Q_2. How will this effect the solution? Each point Q on the x-axis has what is termed



PROPAGATION OF DISTURBANCES

Figure 4

as a range of influence. This is the region of the x,t plane in which the values of u and c are influenced by the initial conditions at Q. This area is defined by the characteristics issuing from Q. Consequently, for the particular case under consideration, the segment $Q_1 Q_2$ will have an influence on u and c for the shaded region in Figure 4. The two curves are given by $C_1: d_x/d_t = u + c$ and $C_2: d_x/d_t = u - c$. u is defined as the horizontal velocity of the moving fluid. The speed of the disturbance as it moves through the flowing water must therefore be given by c. Thus, the validity of the phase speed relation seems apparent.

One final argument can be formulated by the use of the method of characteristics. Although he does not discuss this aspect of his study, Greenspan (1958) outlines this proof. Again, consider the method of characteristics to be formed using a term c defined as $c = [g(\eta + h)]^{\frac{1}{2}}$. A wave which is progressing into quiescent water is considered. Of concern is the forward moving characteristic curve, $\frac{dx}{dt} = u + c$, which contains the wave front. Since the water immediately preceding the wave front is quiescent, the value of u must be zero at the front. The characteristic, and consequently the wave front, must be progressing with speed c. Hence, the phase speed is given by $c = [g(\eta + h)]^{\frac{1}{2}}$.

V. DERIVATION OF WAVE-INDUCED VELOCITY AND WAVE PHASE SPEED USING IWAGAKI AND SAKAI PERTURBATION TECHNIQUE

A. DERIVATION OF $u^{(2)}$

The second order wave induced velocity $u^{(2)}$ is derived using the perturbation technique used by Iwagaki and Sakai to derive a second order surface profile. All terms used are defined in the list of symbols. The initial equations employed are (18) and (19),

$$u_t^{(2)} + u^{(1)} \cdot u_x^{(1)} + g \cdot \eta_x^{(2)} = 0$$
$$\eta_t^{(2)} + \left\{ u^{(1)} \cdot \eta^{(1)} + u^{(2)} h \right\}_x = 0.$$

These equations result when u and η are given as power series expansions of a small quantity α and these expressions substituted into the long wave equations. Eliminating $\eta^{(2)}$ from (18) and (19), substituting $h = i \cdot x$ and grouping terms of $u^{(2)}$ yields

$$u_{tt}^{(2)} - g\left\{ i x\, u_{xx}^{(2)} + 2 i\, u_x^{(2)} \right\} =$$
$$g\left\{ 2 u_x^{(1)} \eta_x^{(1)} + u^{(1)} \eta_{xx}^{(1)} + u_{xx}^{(1)} \eta^{(1)} - u_t^{(1)} u_x^{(1)} - u^{(1)} u_{xt}^{(1)} \right\}. \qquad (30)$$

Using the first order term for u given by Iwagaki and Sakai and differentiating with respect to x gives

$$u_x^{(1)}(x,t) = -a/2\left(\frac{g}{i}\right)^{1/2} x^{-3/2}\{\sin\sigma t\cdot J_1[p(x)] + \cos\sigma t\cdot N_1[p(x)]\}$$

$$+\frac{a\sigma}{ix}\left\{\sin\sigma t\cdot\left(J_0[p(x)] - \frac{1}{p(x)}J_1[p(x)]\right)\right.$$

$$\left. + \cos\sigma t\cdot\left(N_0[p(x)] - \frac{1}{p(x)}N_1[p(x)]\right)\right\} \qquad (31)$$

where $p(x) = 2\sigma\left(\frac{x}{gi}\right)^{1/2}$.

Similarly, differentiating (31) with respect to x provides an expression for $u_{xx}^{(1)}$,

$$u_{xx}^{(1)}(x,t) = a\left(\frac{g}{i}\right)^{1/2} x^{-5/2}\{\sin\sigma t\cdot J_1[p(x)] + \cos\sigma t\cdot N_1[p(x)]\}$$

$$-\frac{2a\sigma}{ix^2}\left\{\sin\sigma t\cdot\left(J_0[p(x)] - \frac{1}{p(x)}J_1[p(x)]\right]\right.$$

$$\left. + \cos\sigma t\cdot\left(N_0[p(x)] - \frac{1}{p(x)}N_1[p(x)]\right)\right\}$$

$$-\frac{a\sigma^2}{i(gi)^{1/2}}x^{-3/2}\{\sin\sigma t\cdot J_1[p(x)] + \cos\sigma t\, N_1[p(x)]\}. \qquad (32)$$

An equation for $u_{xt}^{(1)}$ can be obtained from (31) by differentiating with respect to t,

$$u_{xt}^{(1)}(x,t) = -\frac{a\sigma}{2}\left(\frac{g}{i}\right)^{1/2} x^{-3/2}\{\cos\sigma t\cdot J_1[p(x)] - \sin\sigma t\cdot N_1[p(x)]\}$$

$$+\frac{a\sigma^2}{ix}\left\{\cos\sigma t\cdot\left(J_0[p(x)] - \frac{1}{p(x)}J_1[p(x)]\right]\right.$$

$$\left. - \sin\sigma t\cdot\left(N_0[p(x)] - \frac{1}{p(x)}N_1[p(x)]\right)\right\}. \qquad (33)$$

When $u^{(1)}$ is differentiated with respect to t we obtain

$$u_t^{(1)}(x,t) = a\left(\frac{g}{i}\right)^{1/2}\sigma x^{-\frac{1}{2}}\Big\{\cos\sigma t\cdot J_1[p(x)] - \sin\sigma t\cdot N_1[p(x)]\Big\}. \tag{34}$$

Similarly, differentiating $\eta^{(1)}(x,t)$ yields a set of equations,

$$\eta_x^{(1)}(x,t) = -a\sigma/(gi)^{1/2}\; x^{-1/2}\Big\{\cos\sigma t\cdot J_1[p(x)] - \sin\sigma t\cdot N_1[p(x)]\Big\} \tag{35}$$

and

$$\begin{aligned}\eta_{xx}^{(1)}(x,t) = {} & \frac{a\sigma}{2(gi)^{1/2}}\, x^{-3/2}\Big\{\cos\sigma t\cdot J_1[p(x)] - \sin\sigma t\cdot N_1[p(x)]\Big\} \\ & - \frac{a\sigma^2}{gix}\Big\{\cos\sigma t\cdot\Big(J_0[p(x)] - \frac{1}{p(x)}J_1[p(x)]\Big) \\ & - \sin\sigma t\cdot\Big(N_0[p(x)] - \frac{1}{p(x)}N_1[p(x)]\Big)\Big\}.\end{aligned} \tag{36}$$

Equations (31) through (36) can be used to find expressions for the individual terms on the right hand side of (30). When these expressions are combined, the right side of (30) is evaluated as

$$\begin{aligned}\cos 2\sigma t\,\Big\{ & a^2 g\left(\frac{g}{i}\right)^{1/2} x^{-5/2}\Big(J_0[p(x)]\,N_1[p(x)] + J_1[p(x)]\,N_0[p(x)]\Big) \\ & + \frac{a^2 g\sigma}{i}\, x^{-2}\Big(\frac{11}{2}\,J_1[p(x)]\,N_1[p(x)]\end{aligned}$$

$$-2J_0[p(x)]N_0[p(x)]\Big)$$

$$-\frac{3a^2\sigma^2}{i}\left(\frac{g}{i}\right)x^{-3/2}\Big(J_0[p(x)]N_1[p(x)]$$

$$+J_1[p(x)]N_0[p(x)]\Big)\Big\}$$

$$+\sin 2\sigma t\Big\{a^2 g\left(\frac{g}{i}\right)^{1/2}x^{-5/2}\Big(J_0[p(x)]J_1[p(x)]$$

$$-N_0[p(x)]N_1[p(x)]$$

$$+\frac{a^2 g\sigma}{i}x^{-2}\Big(\frac{11}{4}J_1^2[p(x)]-J_0^2[p(x)]$$

$$+N_0^2[p(x)]-\frac{11}{4}N_1^2[p(x)]\Big)\Big\}$$

$$+a^2 g\left(\frac{g}{i}\right)^{1/2}x^{-5/2}\Big\{J_0[p(x)]N_1[p(x)]-J_1[p(x)]N_0[p(x)]\Big\}. \quad (37)$$

The asymptotic expansions of the Bessel and Neumann (Weber) functions are

$$\left.\begin{aligned} &J_r(w)\sim\left(\frac{2}{\pi w}\right)^{1/2}\cos\left(w-r\pi/2-\pi/4\right)\\ \text{and}\quad &N_r(w)\sim\left(\frac{2}{\pi w}\right)^{1/2}\sin\left(w-r\pi/2-\pi/4\right),\end{aligned}\right\} \quad (38)$$

the approximations being valid for values of w such that $|w| \geq 1.0$.

Substitution of (38) into (37) gives as a final expression for the right side of (30),

$$g\left(2u_x^{(1)}\eta_x^{(1)}+u^{(1)}\eta_{xx}^{(1)}+u_{xx}^{(1)}\eta^{(1)}\right)-u_t^{(1)}-u^{(1)}u_{xt}^{(1)}=$$

$$\cos 2\sigma t\Big\{\frac{3a^2\sigma g}{\pi i}x^{-2}\cos[2\varphi(x)]$$

$$
\begin{aligned}
&- \tfrac{15}{4} \frac{a^2 g}{\pi}\left(\frac{g}{i}\right)^{1/2} x^{-5/2} \sin[2\varphi(x)] \\
&- \frac{a^2 g^2}{\pi\sigma} x^{-3} \cos[2\varphi(x)]\Big\} \\
&+ \sin 2\sigma t \Big\{ \frac{-3a^2 g\sigma}{\pi i} x^{-2} \sin[2\varphi(x)] - \frac{15}{4}\frac{a^2 g}{\pi}\left(\frac{g}{i}\right)^{1/2} x^{-5/2} \cos[2\varphi(x)] \\
&+ \frac{a^2 g^2}{\pi\sigma} x^{-3} \sin[2\varphi(x)] \Big\} \\
&- \frac{a^2 g^2}{\pi\sigma} x^{-3}
\end{aligned}
\tag{39}
$$

where $\varphi(x) = 2\sigma\left(\frac{x}{gi}\right)^{1/2} - \pi/4$.

The solution of $u^{(2)}$ is assumed to be of the form

$$u^{(2)}(x,t) = A(x)\cos 2\sigma t + B(x)\sin 2\sigma t + C(x) . \tag{40}$$

Use of this expression yields equations for the individual terms on the left side of (30),

$$u_x^{(2)}(x,t) = A'(x)\cos 2\sigma t + B'(x)\sin 2\sigma t + C'(x), \tag{41}$$

$$u_{xx}^{(2)}(x,t) = A''(x)\cos 2\sigma t + B''(x)\sin 2\sigma t + C''(x), \tag{42}$$

$$u_t^{(2)}(x,t) = 2\sigma\{B(x)\cos 2\sigma t - A(x)\sin 2\sigma t\}, \tag{43}$$

$$u_{tt}^{(2)}(x,t) = -4\sigma^2\{A(x)\cos 2\sigma t + B(x)\sin 2\sigma t\} . \tag{44}$$

Combination of (41) through (44) can be used to provide an expression for the left side of (30),

$$u_{tt}^{(2)} - g\left(ix\, u_{xx}^{(2)} + 2i\, u_x^{(2)}\right) =$$

$$\cos 2\sigma t\left\{-4\sigma^2 A(x) - gix A''(x) - 2gi A'(x)\right\}$$

$$+ \sin 2\sigma t\left\{-4\sigma^2 B(x) - gix B''(x) - 2gi B'(x)\right\}$$

$$+ \left\{-gix C''(x) - 2gi C'(x)\right\}. \tag{45}$$

Comparison of (39) and (45) allows a determination of A(x), B(x) and C(x) to be made. This is accomplished for A(x) and B(x) by equating the respective coefficients of the sine and cosine terms in the two equations while C(x) is found by equating the terms independent of time. This procedure gives

$$A(x) = \frac{a^2}{\pi i}\left(\frac{g}{i}\right)^{1/2} x^{-3/2} \sin[2\varphi(x)] + 3/5\, \frac{a^2 g}{\pi\sigma i}\, x^{-2} \cos[2\varphi(x)], \tag{46}$$

$$B(x) = \frac{a^2}{\pi i}\left(\frac{g}{i}\right)^{1/2} x^{-3/2} \cos[2\varphi(x)] - 3/5\, \frac{a^2 g}{\pi\sigma i}\, x^{-2} \sin[2\varphi(x)], \tag{47}$$

and

$$C(x) = \frac{a^2 g}{2\pi\sigma i}\, x^{-2}. \tag{48}$$

Substitution of (46), (47) and (48) into the left side of (30) yields

$$
\begin{aligned}
&\cos 2\sigma t \left\{ \frac{3a^2 g\sigma}{\pi i} x^{-2} \cos[2\varphi(x)] \right. \\
&\quad - \frac{15}{4} \frac{a^2 g}{\pi} \left(\frac{g}{i}\right)^{1/2} x^{-5/2} \sin[2\varphi(x)] \\
&\quad \left. - 6/5 \frac{a^2 g^2}{\pi\sigma} x^{-3} \cos[2\varphi(x)] \right\} \\
&+ \sin 2\sigma t \left\{ \frac{-3a^2 g\sigma}{\pi i} x^{-2} \sin[2\varphi(x)] \right. \\
&\quad - 15/4 \frac{a^2 g}{\pi} \left(\frac{g}{i}\right)^{1/2} x^{-5/2} \cos[2\varphi(x)] \\
&\quad \left. + 6/5 \frac{a^2 g^2}{\pi\sigma} x^{-3} \sin[2\varphi(x)] \right\} \\
&+ \frac{a^2 g^2}{\pi\sigma} x^{-3} . \qquad (49)
\end{aligned}
$$

Similar to the development of Iwagaki and Sakai for $\eta^{(2)}$, the coefficients A(x), B(x) and C(x) which have been deduced do not yield an exact solution of (30). Comparison of (39) and (49) shows that discrepancies exist in the third term coefficients of cos $2\sigma t$ and sin $2\sigma t$ (i.e., 6/5 versus 1.0). The three terms which comprise the coefficients of cos $2\sigma t$ and sin $2\sigma t$ involve x^{-2}, $x^{-5/2}$ and x^{-3}. Use of $T = \frac{2\pi}{\sigma}$ and $h = i \cdot x$ enables the ratio of the third term to the first term to be evaluated as

$$
\text{3rd term/1st term} \sim \frac{Tgi}{2\pi(gh)^{1/2}} . \qquad (50)
$$

It is noted that the first and third term coefficients are in phase. For the case of i = 1/10, h = 25 cm and T = 3 seconds, (50) indicates that the ratio of the third term to the first is equal to 0.089. Hence, the difference between the coefficients created by the use of (46), (47) and (48) is negligible and (40) may therefore be evaluated as

$$u^{(2)}(x,t) = \cos 2\sigma t \left\{ \frac{a^2}{\pi i} \left(\frac{g}{i}\right)^{1/2} x^{-3/2} \sin[2\varphi(x)] + 3/5 \frac{a^2 g}{\pi i \sigma} x^{-2} \cos[2\varphi(x)] \right\}$$

$$+ \sin 2\sigma t \left\{ \frac{a^2}{\pi i} (g/i)^{1/2} x^{-3/2} \cos[2\varphi(x)] - 3/5 \frac{a^2 g}{\pi i \sigma} x^{-2} \sin[2\varphi(x)] \right.$$

$$+ \frac{a^2 g}{2\pi \sigma i} x^{-2} . \tag{51}$$

Equation (51) can be simplified to

$$u^{(2)}(x,t) = \frac{-a^2}{\pi i} (g/i)^{1/2} x^{-3/2} \cos\left\{ 2\left[\sigma t + 2\sigma \left(\frac{x}{gi}\right)^{1/2} - \frac{\pi}{4}\right] \right.$$

$$\left. + \pi/2 + \tan^{-1}\left[\frac{3}{5} \frac{(gi)^{1/2}}{\sigma} x^{-1/2}\right] \right\} . \tag{52}$$

In order to make this simplification, it is necessary to assume that

$$\left[9/10 \frac{gi}{\sigma x} + 1 \right]^{1/2} \simeq 1.0 . \tag{53}$$

The applicable range of solution for this perturbation scheme, which is discussed in Section V.C., restricts the computations to regions where $2\sigma(\frac{x}{gi})^{\frac{1}{2}} \geq 1.0$. If the lower limit of this relation, i.e. $2\sigma(\frac{x}{gi})^{\frac{1}{2}} = 1.0$, is substituted into the left side of (53), a numerical result of 1.166 is obtained. This error appears to be rather significant. Utilizing a limit of $2\sigma \left(\frac{x}{gi}\right)^{\frac{1}{2}} \geq 4.0$, the left side of (53) is evaluated as 1.044. Consequently, the use of this ratio as a limit on the range of applicability of the solution may seem appropriate. Further discussion of this parameter is presented in Section V.C.

B. SECOND ORDER EQUATIONS FOR η AND u

Originally, it was assumed that η and u could be expressed as

$$\eta = \alpha \eta^{(1)} + \alpha^2 \eta^{(2)} + \cdots$$

and

$$u = \alpha u^{(1)} + \alpha^2 u^{(2)} + \cdots$$

It is therefore necessary to combine the expressions for $\eta^{(1)}$ and $\eta^{(2)}$ and for $u^{(1)}$ and $u^{(2)}$ to determine the final relations. It is noted that for $\eta(x,t)$, Iwagaki and Sakai neglected the second part of (22) which is independent of t. They felt justified in so doing since these terms effect only the stillwater depth. Their study was concerned with wave heights and profiles and consequently the stillwater depth was not required. This simplification is not valid for the present study since the phase speed, given by $c = [g(\eta + h)]^{\frac{1}{2}}$, is effected by the stillwater depth. Furthermore, the ratio of the coefficients of the first term independent of t to the first term on the right side of (22) is $\left[\frac{T}{2\pi} \left(\frac{g}{h}\right)^{\frac{1}{2}} i\right]^{1}$. Utilizing the upper limit for $T\left(\frac{g}{h}\right)^{\frac{1}{2}}$ of $4\pi/i$ given by Iwagaki and Sakai, the importance of including these terms is apparent. (The upper limit of $T\left(\frac{g}{h}\right)^{\frac{1}{2}}$ is discussed in Section V.C.)

When the first order equations for u and η given by Iwagaki and Sakai, equations (20) and (21), are expressed using the asymptotic approximations for the Bessel and Weber functions the following results are obtained,

$$\eta^{(1)}(x,t) = a\left[\frac{(gi)^{1/2}}{\pi\sigma}\right]^{1/2} x^{-1/4}\cos\left\{\sigma t + 2\sigma\left(\frac{x}{gi}\right)^{1/2} - \pi/4\right\} \tag{54}$$

and

$$u^{(1)}(x,t) = -a\left(g/i\right)^{1/4}\left(g/\pi\sigma\right)^{1/2} x^{-3/4}\cos\left\{\sigma t + 2\sigma\left(\frac{x}{gi}\right)^{1/2} - \pi/4\right\}. \tag{55}$$

Equations (22), (52), (54) and (55) can now be used to arrive at second order expressions for η and u,

$$\begin{aligned}\eta(x,t) = {} & \alpha\left[a\left\{\frac{(gi)^{1/2}}{\pi\sigma}\right\}^{1/2} x^{-1/4}\cos\left\{\sigma t + 2\sigma\left(\frac{x}{gi}\right)^{1/2} - \pi/4\right\}\right] \\ & + \alpha^2\left[\frac{a^2}{\pi i} x^{-1}\cos\left\{2\left[\sigma t + 2\sigma\left(\frac{x}{gi}\right)^{1/2} - \frac{\pi}{4}\right] + \frac{\pi}{2} + \tan^{-1}\left(\frac{3}{10}\,\frac{(gi)^{1/2}}{\sigma}\,x^{-1/2}\right)\right\}\right. \\ & + \frac{a^2}{4\pi\sigma}\left(g/i\right)^{1/2} x^{-3/2}\cos\left\{2\left[2\sigma\left(\frac{x}{gi}\right)^{1/2} - \pi/4\right]\right\} \\ & \left. - \frac{a^2 g}{16\pi\sigma^2} x^{-2}\sin\left\{2\left[2\sigma\left(\frac{x}{gi}\right)^{1/2} - \frac{\pi}{4}\right]\right\}\right]\end{aligned} \tag{56}$$

and

$$u(x,t) = \alpha\left[-a\left(g/i\right)^{1/4}\left(\frac{g}{\pi\sigma}\right)^{1/2} x^{-3/4}\cos\left\{\sigma t + 2\sigma\left(\frac{x}{gi}\right)^{1/2} - \pi/4\right\}\right]$$
$$+\alpha^2\left[-\frac{a^2}{\pi i}\left(\frac{g}{i}\right)^{1/2} x^{-3/2}\cos\left[2\left\{\sigma t + 2\sigma\left(\frac{x}{gi}\right)^{1/2} - \frac{\pi}{4}\right\} + \pi/2 + \tan^{-1}\left\{\frac{3}{5}\frac{(gi)^{1/2}}{\sigma}x^{-1/2}\right\}\right]\right]. \qquad (57)$$

C. DETERMINATION OF THE PERTURBATION PARAMETER,

The next question which must be addressed is how should the perturbation parameter, α , be defined. The first step in this determination, as outlined by Iwagaki and Sakai, is to formulate a region of applicability for the solution. In their study, they begin by considering a wave celerity of $c = (gh)^{\frac{1}{2}}$. This is cause for some concern in that it differs from the accepted expression of $c = [g(\eta+h)]^{\frac{1}{2}}$. The problem, however, is to find a region of water where the theory can be applied; hence, use of an average velocity for the entire area seems appropriate. Prior to the arrival of the wave form and after its passage, the surface is at the stillwater depth and $\eta = 0$. Therefore, $\eta = 0$ provides an average value for η over the region and $c = (gh)^{\frac{1}{2}}$ gives a representative average speed. In addition, over a large portion of the region η will be small compared to h. Ayyar's (1972) calculations indicate that even at the extreme point of breaking, the phase speed at the crest will only vary from this value by a factor of $(2)^{\frac{1}{2}}$.

Using this value for c, Iwagaki and Sakai derive the relation

$$h/L = \frac{1}{T(g/h)^{1/2}} \quad . \tag{58}$$

The authors conclude from this that an upper limit on h/L gives a lower limit for $T(g/h)^{\frac{1}{2}}$. They then considered establishing an upper limit for $T(g/h)^{\frac{1}{2}}$. For this purpose they recalled that the approximations used for the Bessel and Neumann functions required that

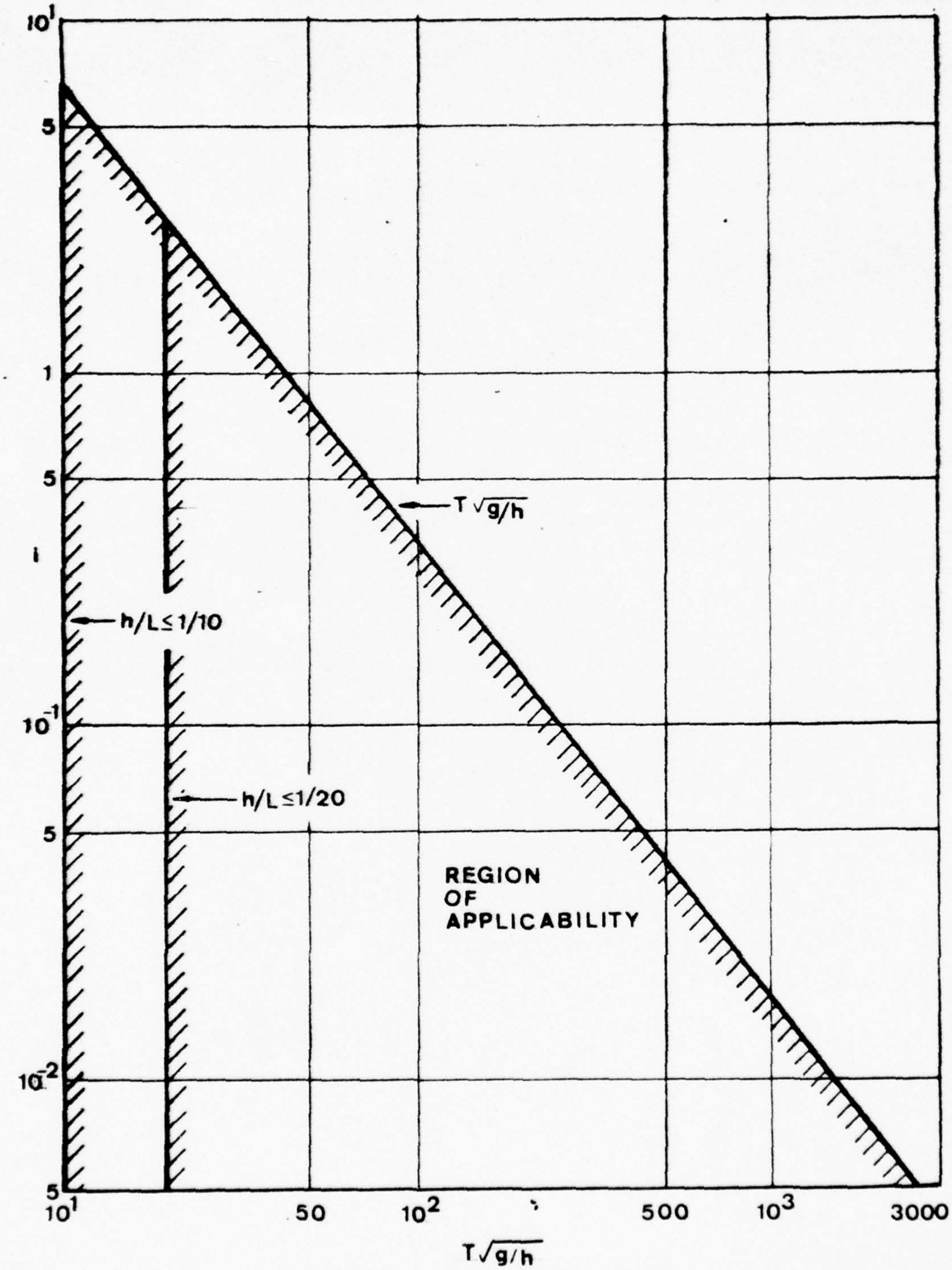
$$2\sigma\left(\frac{x}{gi}\right)^{1/2} \geq 1.0. \tag{59}$$

This in turn, implies that

$$T(g/h)^{1/2} \leq 4\pi/i \; . \tag{60}$$

Thus, (60) defines the upper limit of $T(g/h)^{\frac{1}{2}}$. Figure 5 shows how (58) and (60) are combined to define the region of applicability for the solution.

Iwagaki and Sakai select as the perturbation parameter the ratio h_1/L_o, where h_1 is the largest depth in the applicable range of the solution and L_o is the deep water



APPLICABLE RANGE OF SOLUTION

Figure 5

wave length given by $L_o = gT^2/2\pi$. Use of this ratio may at first seem rather arbitrary. The motivation for this choice becomes more apparent when the derivation of the long wave equations by Freidrichs is recalled. His perturbation parameter involved a representative length and depth quantity for the wave. Consequently, a similar ratio seems logical when considering the small value of α.

The ease with which this ratio may be evaluated provides an additional incentive for its selection; examination of the range of applicability of the solution is all that is required to obtain its value. An additional comment about this region is required. The upper limit to be placed upon h/L has not been previously discussed. In fact, two values for this parameter are shown in Figure 5. For their numerical calculations, Iwagaki and Sakai use $h/L \leq 1/20$. The selection was arbitrary but conforms to general usage in wave theory. The theory was developed for shallow water where the pressure is hydrostatic; for this region, $h/L \leq 1/20$ provides a reasonable limit. When this value is assumed, (58) requires that

$$T\left(g/h\right)^{1/2} \geq 20. \tag{61}$$

Applying $L_o = gT^2/2\pi$ to (61),

$$h_{\perp}/L_o = 0.0157. \tag{62}$$

Equation (62) will be adopted for the value of α throughout the majority of this presentation.

Iwagaki and Sakai discuss the fact that use of (62) restricts the discussion to waves for which $H_o/L_o \doteq 0.006$. For greater H_o/L_o values, the theoretical energy flux curves predict that the deep water waves will break prior to arriving at the point where $h_1/L_o = 0.0157$. In order to extend the theory to situations where H_o/L_o exceeds 0.006, it is necessary to increase the upper limit placed upon h/L. Several cases for which $h/L \leqq 1/15$ are investigated in this study to determine the applicability of the theory to a domain of H_o/L_o values greater than 0.006.

A final comment may now be made concerning the approximation of (53). Use of (59) evaluates (60) as 1.16, not the assumed value 1.0. However, when these values are multiplied by the quantity α^2, as is required by (56) and (57), the difference becomes negligibly small. Hence the determination by Iwagaki and Sakai for the limit $2\sigma(\frac{x}{gi})^{\frac{1}{2}} \geq 1.0$ seems appropriate.

D. SECOND ORDER EXPRESSION FOR c

A second order expression for the phase speed c can be found by substituting (56) into the phase speed relation, $c = [g(\eta + h)]^{\frac{1}{2}}$, giving

$$
\begin{aligned}
c = -\Big[& \alpha \Big[a g \frac{(gi)^{1/4}}{(\pi\sigma)^{1/2}} x^{-1/4} \cos\Big\{ \sigma t + 2\sigma \left(\frac{x}{gi}\right)^{1/2} - \pi/4 \Big\} \Big] \\
& + \alpha^2 \Big[\frac{a^2 g}{\pi i} x^{-1} \cos\Big\{ 2\Big[\sigma t + 2\sigma\left(\frac{x}{gi}\right)^{1/2} - \frac{\pi}{4}\Big] + \pi/2 + \tan^{-1}\Big[3/10 \, \frac{(gi)^{1/2}}{\sigma} x^{-1/2} \Big] \Big\} \\
& + \frac{a^2 g^{3/2}}{4\pi\sigma i^{1/2}} x^{-3/2} \cos\Big\{ 2\Big[2\sigma \left(\frac{x}{gi}\right)^{1/2} - \frac{\pi}{4} \Big] \Big\} \\
& - \frac{a^2 g^2}{16\pi\sigma^2} x^{-2} \sin\Big\{ 2\Big[2\sigma \left(\frac{x}{gi}\right)^{1/2} - \frac{\pi}{4} \Big] \Big\} \Big] \\
& + g i x \Big]^{1/2} .
\end{aligned}
\tag{63}
$$

The negative square root is utilized due to the fact that the wave is progressing in the negative x-direction.

E. DETERMINATION OF a

Prior to applying the kinematic breaking criterion to the preceding equations, the value of the term a in (56), (57) and (63) must be determined. This evaluation follows closely that which was outlined by Iwagaki and Sakai. As mentioned previously, these authors neglected those terms in η which were independent of t. Although these terms are included for the determination of c, they can be eliminated for the purpose of establishing the value of a. The validity of this simplification stems from the fact that the value of the constant will be evaluated at the point where $h = h_1$, the deepest depth in the applicable range of the solution. Examination of the terms independent of time in (56) shows that they decrease in absolute value for increasing x; the terms becoming negligibly small at the point where $h = h_1$. Consequently, for the determination of a, η will be assumed to be expressed by

$$\eta(x,t) = \alpha\left[a\left\{\frac{(gi)^{1/2}}{\pi\sigma}\right\}^{1/2} x^{-1/4} \cos\left\{\sigma t + 2\sigma\left(\frac{x}{gi}\right)^{1/2} - \frac{\pi}{4}\right\}\right]$$
$$+ \alpha^2\left[\frac{a^2}{\pi i} x^{-1} \cos\left\{2\left[\sigma t + 2\sigma\left(\frac{x}{gi}\right)^{1/2} - \frac{\pi}{4}\right]\right.\right.$$
$$\left.\left. + \frac{\pi}{2} + \tan^{-1}\left[\frac{3}{10} \frac{(gi)^{1/2}}{\sigma} x^{-\frac{1}{2}}\right]\right\}\right] \qquad (64)$$

Substituting $\alpha = h_1/L_o$ into (64) yields

$$\eta/h_1 = A^{(1)} \cos\theta + A^{(2)} \cos(2\theta + \delta) \qquad (65)$$

where

$$\left.\begin{aligned}
A^{(1)} &= 2^{-1/4}\pi^{-3/4} i^{1/2} (h_1/L_o)^{3/4}(h_1/h)^{1/4}(a/h_1), \\
A^{(2)} &= \pi^{-1}(h_1/L_o)^2 (h_1/h)(a/h_1)^2, \\
\theta &= (2\pi/T)t + 2\pi\left[(2/\pi)^{1/2}(h/L_o)^{-1/2}\right]\left(\frac{x}{L_o}\right) \\
&\quad - \pi/4 \\
\delta &= \pi/2 + \tan^{-1}\left\{\frac{3}{10}(2\pi)^{-1/2} i (h/L_o)^{-1/2}\right\}.
\end{aligned}\right\} \qquad (66)$$

If $A_1^{(1)}$ and $A_1^{(2)}$ are used to denote $A^{(1)}$ and $A^{(2)}$ at $h = h_1$, then from (66)

$$\left.\begin{aligned} A_1^{(1)} &= 2^{-1/4}\pi^{-3/4} i^{-1/2}(h_1/L_o)^{3/4} a/h_1, \\ A_1^{(2)} &= \pi^{-1}(h/L_o)^2 (a/h_1)^2. \end{aligned}\right\} \tag{67}$$

In addition, at $h = h_1$, $\tan^{-1}\left(\frac{3}{10}(2\pi)^{-\frac{1}{2}} i(h/L_o)^{-\frac{1}{2}}\right)$ becomes negligibly small compared to $\pi/2$ and can therefore be neglected.

The wave profile at $h = h_1$, represented by η_1, is now given by,

$$\eta_1/h = A_1^{(1)} f(\theta) \tag{68}$$

where

$$\left.\begin{aligned} f(\theta) &= \cos\theta - b\sin 2\theta, \\ b &= A_1^{(2)}/A_1^{(1)}. \end{aligned}\right\} \tag{69}$$

At $h = h_1$, the wave height is assumed to be twice the amplitude, η_1, hence,

$$H_1/h_1 = 2A_1^{(1)} f(\theta_c) \tag{70}$$

where θc is evaluated from $df/d\theta = 0$. Iwagaki and Sakai determine θc as

$$\theta_c = \arcsin\left\{\left[1/4b - \left(1/(16b^2)+2\right)^{1/2}\right]/2\right\}. \tag{71}$$

The problem now is to find a value for H_1/h_1. Use of the theoretical curves for wave height change, which are based upon wave energy flux, provides a value for H_1/H_o when h_1/L_o is known. H_o/L_o, the ratio of the deep water wave height to the deep water wave length, can be determined for various wave conditions and then H_1/h_1 is found through the identity

$$H_1/h_1 = (H_1/H_o)(H_o/L_o)/(h_1/L_o). \tag{72}$$

Equations (67), (69), (71) and (72) can now be substituted into (70) to evaluate a. Simple computer techniques, as discussed in Appendix B, provide a determination of a for specific values of i, H_o/L_o and period T.

VI. BREAKING CRITERION DERIVED

A. APPLICATION OF KINEMATIC BREAKING CRITERION

The derived horizontal wave induced velocity and wave phase speed are used to derive a breaking criterion. As stipulated by the kinematic breaking criterion, breaking will occur when the horizontal particle velocity equals the phase speed velocity. Theoretically, this condition may exist for several points in space and time. Of concern, however, is the specific case for which the horizontal distance from the beach at which u equals c is maximized. This will be the first position at which the approaching wave may break and hence all other cases are purely imaginary. The numerical solution technique employed is relatively simple. Subtracting the second order relation for u, (57), from that for the phase speed c, (63), equating the resulting expression to zero, and solving for x and t gives points in space and time at which the kinematic breaking criterion is satisfied. Examination of this solution set yields the maximum horizontal distance at which u equals c. The computer techniques employed are discussed in Appendix B.

B. RESULTS

The numerical results obtained are summarized in Table 1. Several wave conditions are investigated in which the values for the beach slope, wave period, and deep water wave height to deep water wave length (H_o/L_o) are varied. The specific selection of 0.119298 for the beach slope and 8.6 seconds for the period was made to conform with a future study (Hulstrand, 1976) in which experimental data will be used to verify these theoretical results. All other choices are strictly arbitrary.

The first result of interest is that of the ratio η_b/L_o, where η_b is the free surface elevation at the point of breaking. Table 1 shows that for each combination of beach slope and H_o/L_o in the second order solution, the ratio is essentially constant (some small variations occur in the second significant figure). The value of the ratio is independent of the wave period. The consistency of the ratio suggests the use of this parameter as a breaking criteria for specific beach slopes and H_o/L_o conditions.

A second parameter listed in Table 1, that of h_b/H_o, where h_b is the depth at breaking referenced to the still-water level, has often been utilized in the measurement of breaking waves. The theory under investigation yields consistent results for this ratio for specific beach slopes

TABLE 1

Numerical Results of Derived Shallow Water Breaking Criteria

Order of Solution	$h/L \leq$	Beach Slope	H_o/L_o	Period (secs)	y_b/h_b	η_b/L_o	h_b/H_o
2nd	1/20	0.119298	0.001	8.6	2.79452	0.00179	1.00000
2nd	1/20	0.119298	0.001	10.0	2.79718	0.00180	1.00000
2nd	1/20	0.119298	0.001	15.0	2.78646	0.00179	1.00000
2nd	1/20	0.119298	0.002	8.6	2.37132	0.00270	0.98500
2nd	1/20	0.119298	0.002	10.0	2.39132	0.00274	1.01523
2nd	1/20	0.119298	0.002	15.0	2.38360	0.00273	0.99999
2nd	1/20	0.119298	0.004	8.6	2.38848	0.00486	0.87500
2nd	1/20	0.119298	0.004	10.0	2.39192	0.00486	0.87250
2nd	1/20	0.119298	0.004	15.0	2.39071	0.00486	0.87250
2nd	1/20	0.05	0.001	8.6	2.50141	0.00222	1.48000
2nd	1/20	0.05	0.001	10.0	2.52461	0.00226	1.48000
2nd	1/20	0.05	0.001	15.0	2.52625	0.00226	1.48000
2nd	1/20	0.05	0.002	8.6	2.34344	0.00354	1.32000
2nd	1/20	0.05	0.002	10.0	2.32716	0.00353	1.33000
2nd	1/20	0.05	0.002	15.0	2.33129	0.00351	1.32000
2nd	1/20	0.05	0.003	8.6	2.39718	0.00493	1.17652
2nd	1/20	0.05	0.003	10.0	2.40691	0.00497	1.17666
2nd	1/20	0.05	0.003	15.0	2.39442	0.00492	1.17666

Table 1 (Cont'd)

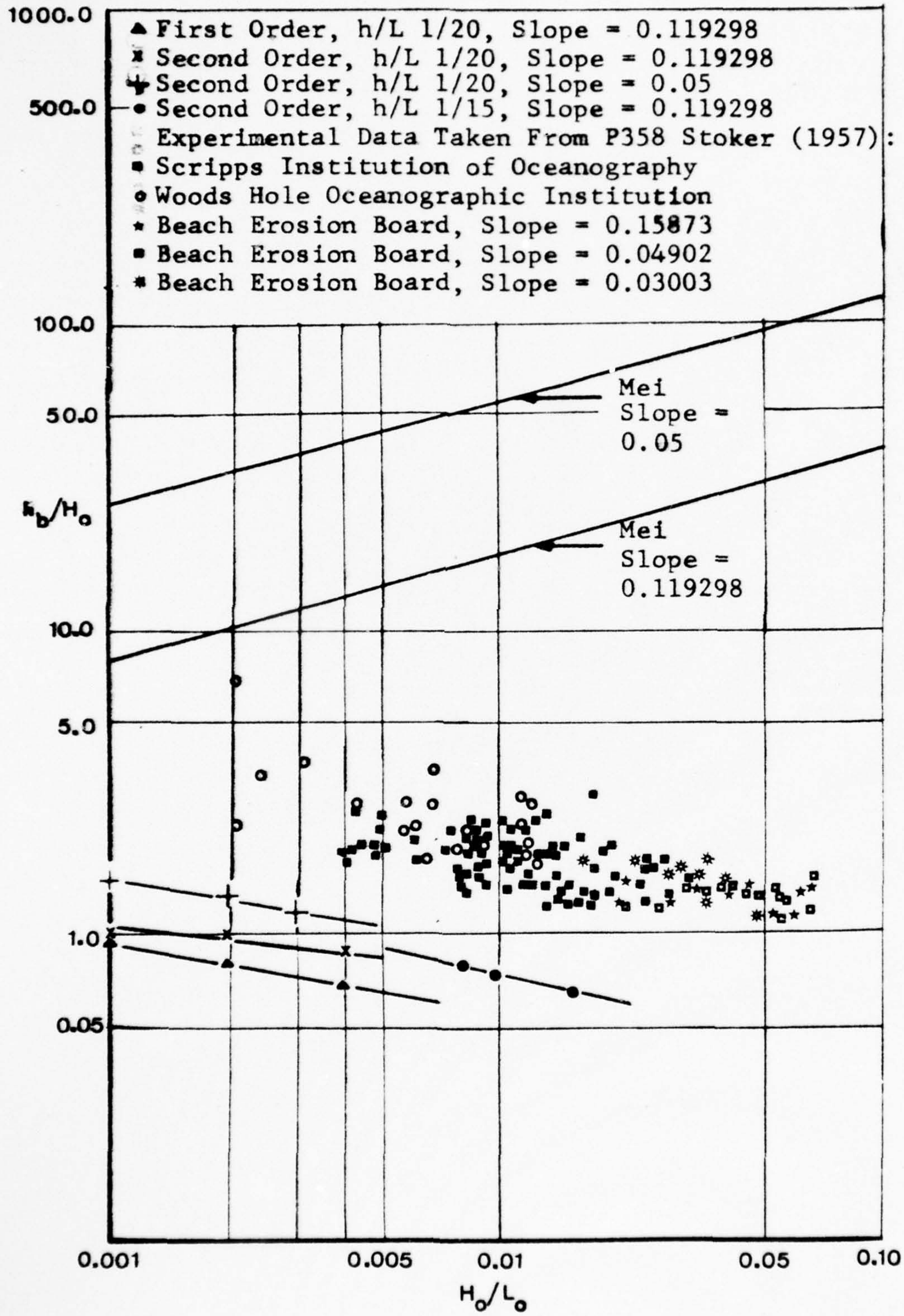
Order of Solution	$h/L \leq$	Beach Slope	H_o/L_o	Period (secs)	y_b/h_b	η_b/L_o	h_b/H_o
2nd	1/15	0.119298	0.008	10.0	2.77502	0.01124	0.79125
2nd	1/15	0.119298	0.008	15.0	2.78116	0.01128	0.79125
2nd	1/15	0.119298	0.010	8.6	2.86780	0.01378	0.73800
2nd	1/15	0.119298	0.015	8.6	2.82185	0.01789	0.65467
1st	1/20	0.119298	0.001	8.6	2.20105	0.00113	0.94000
1st	1/20	0.119298	0.002	8.6	2.33403	0.00209	0.78500
1st	1/20	0.119298	0.004	8.6	2.43484	0.00379	0.66000

and H_o/L_o combinations. Mei (1966) derived a first order solution for this parameter of

$$h_b/H_o \simeq (32\pi)^{1/3}(H_o/L_o)^{1/3}(i)^{-4/3}.$$

Examination of Figure 6 shows that Mei's calculations do not compare favorably with experimental data. Mei attributed this difference to the fact that his solution was confined to first order theory. Figure 6 shows that the theory presented here yields a much closer approximation to the observed experimental data. It is noted that much of the observed data has been accumulated for values of H_o/L_o greater than those applicable to the current calculations. Plots for $h/L \leq 1/20$ and $h/L \leq 1/15$ have been extended through the theoretically derived points for purposes of comparison. Within the concurrent regions of applicability, the theoretically derived values for the various limiting values of h/L are similar. This suggests that the extension of these graphs into the domain of greater H_o/L_o values provides at least an indication of the breaking criteria which would be derived by utilizing h/L ratios applicable for these regions.

The theoretically derived values of h_b/H_o decrease with increasing values of H_o/L_o. This conforms with the trend displayed by the field measurements. In contrast,



1000.0
500.0
100.0
50.0
10.0
5.0
1.0
0.05
h_b/H_o
First Order, h/L 1/20, Slope = 0.119298
Second Order, h/L 1/20, Slope = 0.119298
Second Order, h/L 1/20, Slope = 0.05
Second Order, h/L 1/15, Slope = 0.119298
Experimental Data Taken From P358 Stoker (1957):
Scripps Institution of Oceanography
Woods Hole Oceanographic Institution
Beach Erosion Board, Slope = 0.15873
Beach Erosion Board, Slope = 0.04902
Beach Erosion Board, Slope = 0.03003
Mei
Slope =
0.05
Mei
Slope =
0.119298
0.001
0.005
0.01
0.05
0.10
H_o/L_o

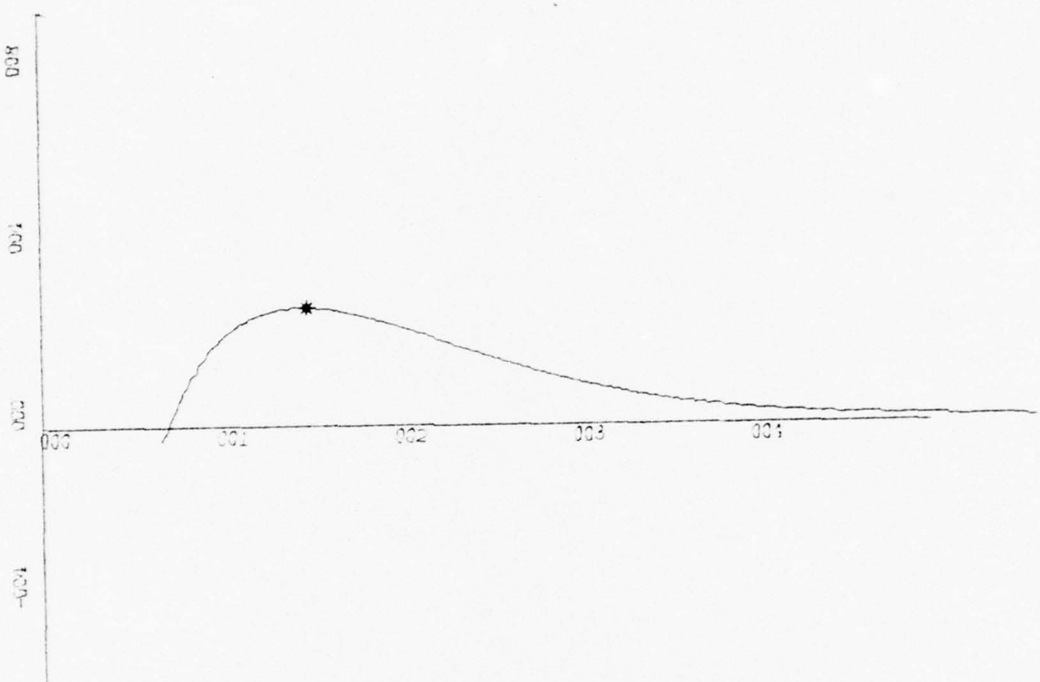
Figure 6

Mei's results show an increase in the ratio with increasing H_o/L_o.

Of concern is the fact the current theory predicts breaking depths less than those observed in the experimental data. Table 1 and Figure 6 include data for the first order solution resulting from the theory presented. Figure 6 shows that the extension to the second order solution produces results which compare more favorably to the experimental results. This suggests that higher order solutions would yield improved results. An additional source of error present in this study results from the fact that the vertical water particle accelerations have been neglected.

A final comment can be made concerning Mei's calculations. The first order solution which is derived from the procedure utilized in this study yields considerably improved results over those of Mei. Therefore Mei's descrepancies cannot be attributed entirely to the restriction to the lowest order solution. The source of error may possibly be due to the use of the geometrical breaking criterion.

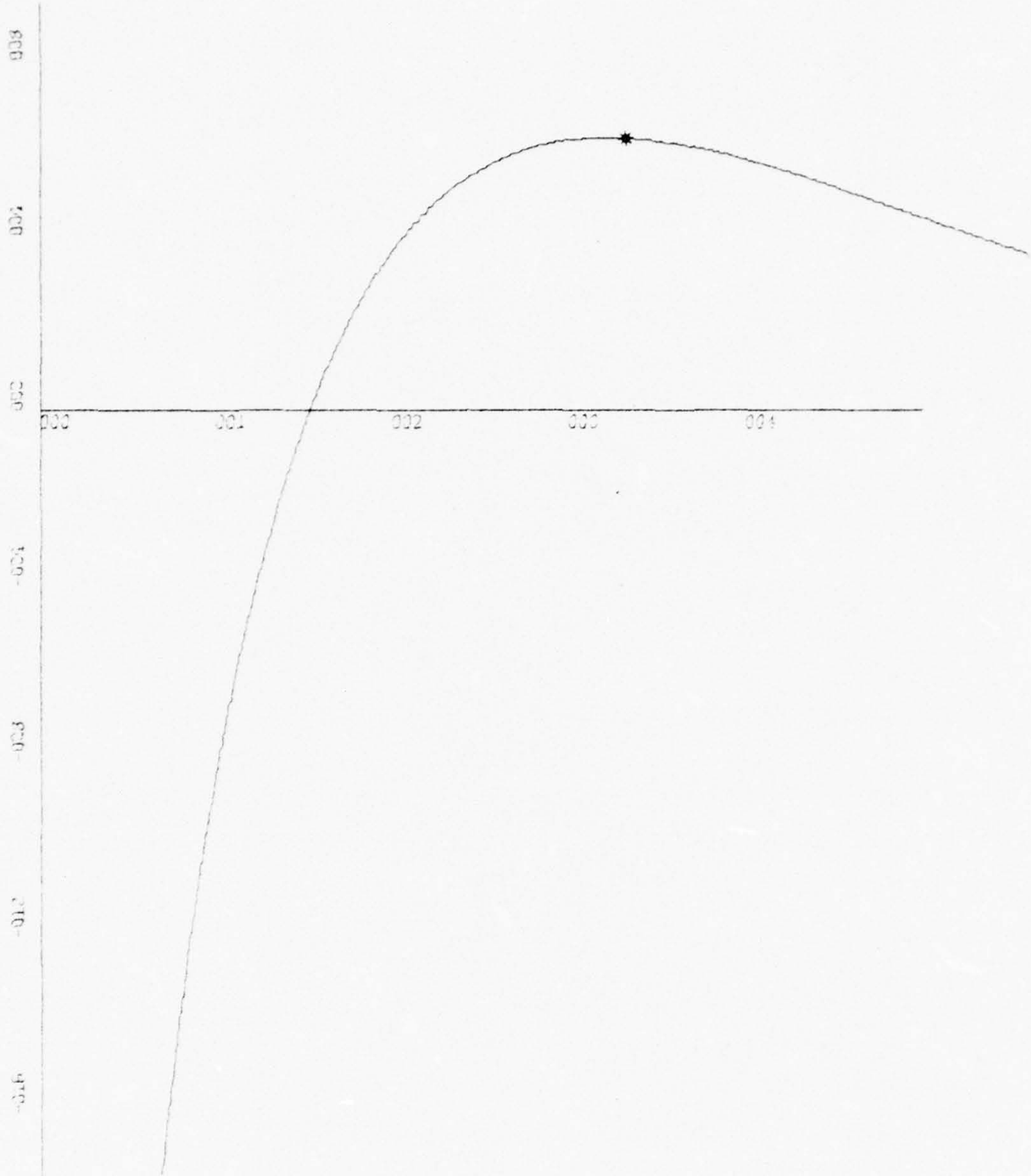
Figures 7 and 8 depict representative wave profiles at breaking for the current theory. These indicate that breaking occurs prior to the attainment of a vertical surface slope. The figures also show that the predicted



X-SCALE=1.00E+01 UNITS INCH.
Y-SCALE=4.00E+00 UNITS INCH.
BREAKING WAVE PROFILE

$h/L \doteq 1/20$
Slope = 0.119298
H_o/L_o = 0.004
Period = 10.0 seconds
* = Predicted Point of Breaking

Figure 7



X-SCALE=1.00E+01 UNITS INCH.
Y-SCALE=4.00E+00 UNITS INCH.

BREAKING WAVE PROFILE

$h/L \leq 1/20$
$H_o/L_o = 0.004$
Slope = 0.119298
Period = 15.0 seconds
* = Predicted Point of Breaking

Figure 8

breaking occurs at the wave crest. Figure 8 shows an unrealistically deep trough in front of the crest. The use of the Bessel function approximation has been shown to be valid for $2\sigma\left(\frac{x}{gi}\right)^{1/2} \geq 1.0$. Substitution into this relation of the values of the terms used to construct Figure 8 requires $x \geq 21.89$ feet. It is seen, therefore, that the excessively deep trough is predicted in a region where the Bessel function approximation is not valid. The predicted breaking for the specific instance shown in the figure is at x = 33.73 feet, which is in the applicable region for the approximation. For each case investigated, the predicted breaking point occurred well within the region where the approximation is accurate; hence the derived breaking criteria is deemed valid.

Ayyar (1970), utilizing the kinematic breaking criterion, produced the additional shallow water breaking criterion of y_b/h_b=2.0. Table 1 summarizes the values obtained in the present study for this ratio, all of which are considerably larger than 2.0. Ayyar's ratio does not account for the variations associated with H_o/L_o suggested by the observed h_b/H_o data. In addition, his theory is limited to waves which have the geometry of a plunging breaker. Ayyar also assumes that breaking occurs at the wave front, which may not in fact take place. The limitations placed upon Ayyar's formulation and the random values of the ratio displayed in Table 1 prevents this from being considered a valid indication of breaking.

The most significant disadvantage associated with the breaking criteria derived from the theory presented in this study is that the ratios require re-calculation for changes in beach slope and deep water wave height to wave length ratio. This objection is similar to that which was raised against Stoker's use of the method of characteristics. The complex dependence of the values of a, u and c upon both the beach slope and H_o/L_o, however, leaves the investigator with little choice but to resort to a numerical solution. In defense of the approach, the solution technique applied to specific situations is relatively simple and requires minimal computations once the beach slope and H_o/L_o are known. Selected data points may be used to construct graphs which approximate the h_b/H_o ratio for each specific beach slope. This can be used to provide an indication of breaking for varying H_o/L_o values.

VII. CONCLUSIONS

Two shallow water breaking criteria have been formulated through the application of the kinematic breaking criterion to a second order solution of the long wave equations. These are the ratios of η_b/L_o and h_b/H_o. Both of these criteria are dependent only upon the beach slope and the ratio of the deep water wave height to the deep water wave length. Each ratio requires re-calculation as these two parameters vary. Comparison with previous theory indicates that the theoretically derived values for h_b/H_o offer significantly improved approximations to the assembled experimental data. The increased accuracy is partially attributed to the use of a higher order solution to the long wave equations. First order solutions obtained indicate that the solution technique applied offers improvement over previous theory. The predicted breaking depths are somewhat less than those observed in field measurements. It is believed that the extension of the theory to higher order solutions of the long wave equations would reduce the error associated with the predicted breaking depth. Inclusion of vertical water particle accelerations would also increase the accuracy of the solution.

APPENDIX A

DERIVATION OF $\eta^{(2)}$

Iwagaki and Sakai have obtained a second order solution for the free surface, $\eta(x,t)$. The relation derived compared favorably with experimental data. Several approximations made in the solution, however, warrant discussion.

Combining equations (18) and (19) so that $u^{(2)}$ is eliminated, gives

$$\eta^{(2)}_{tt} - g\left[\eta^{(2)}_{xx}\, i x + \eta^{(2)}_{x}\, i\right] =$$

$$-\left[u^{(1)}\eta^{(1)}\right]_{xt} + \left[u^{(1)}u^{(1)}_{x}\right]_{x} i x + \left[u^{(1)}u^{(1)}_{x}\right] i \;. \qquad \text{(A-1)}$$

Iwagaki and Sakai substituted their first order equations (20) and (21) into the right side of (A-1). They then offered approximations for the Bessel and Neumann (Weber) functions of

$$J_r(w) \sim \left(2/\pi w\right)^{1/2} \cos\left(w - r\pi/2 - \pi/4\right) \qquad \text{(A-2)}$$

and

$$N_r(w) \sim \left(2/\pi w\right)^{1/2} \sin\left(w - r\pi/2 - \pi/4\right). \qquad \text{(A-3)}$$

The authors provide evidence which shows that for $|w| \geq 1.0$, these asymptotic expansions are accurate. Defining

$\varphi(x) = (2\sigma(x/gi)^{\frac{1}{2}} - \pi/4)$, use of (A-2) and (A-3) allows the right side of (A-1) to be expressed as

$$\cos 2\sigma t \Big\{ -\frac{3a^2\sigma}{\pi}\left(\frac{g}{i}\right)^{1/2} x^{-3/2} \cos[2\varphi(x)] + 5/2\, a^2 g/\pi\; x^{-2} \sin[2\varphi(x)] + \frac{a^2 g}{\pi\sigma}(gi)^{1/2} x^{-5/2}\cos[2\varphi(x)]\Big\}$$

$$+ \sin 2\sigma t\Big\{ \frac{3a^2\sigma}{\pi}\left(\frac{g}{i}\right)^{1/2} x^{-3/2} \sin[2\varphi(x)] + 5/2\, a^2 g/\pi\; x^{-2}\cos[2\varphi(x)] - \frac{a^2 g}{\pi\sigma}(gi)^{1/2} x^{-5/2} \sin[2\varphi(x)]\Big\}$$

$$+\Big\{\frac{a^2\sigma}{\pi}\left(\frac{g}{i}\right)^{1/2} x^{-3/2}\cos[2\varphi(x)] - \frac{3}{2}\, a^2 \frac{g}{\pi}\, x^{-2}\sin[2\varphi(x)]$$

$$- \frac{a^2 g}{\pi\sigma}(gi)^{1/2} x^{-5/2}\cos[2\varphi(x)]. \qquad \text{(A-4)}$$

The solution of $\eta^{(2)}$ was ~~assumed to be~~

$$\eta^{(2)}(x,t) = A(x)\cos 2\sigma t + B(x)\sin 2\sigma t + C(x).$$

This expression was substituted into the left side of equation (A-1) and the result compared to (A-4). From this comparison, the coefficients A(x), B(x) and C(x) are determined as

$$A(x) = -\frac{a^2}{\pi i}\, x^{-1}\sin[2\varphi(x)] - \frac{3}{10}\frac{a^2}{\pi\sigma}\left(\frac{g}{i}\right)^{1/2} x^{-3/2}\cos[2\varphi(x)], \qquad \text{(A-5)}$$

$$B(x) = -\frac{a^2}{\pi i} x^{-1} \cos[2\varphi(x)] + \frac{3}{10}\frac{a^2}{\pi i}\left(\frac{g}{i}\right)^{1/2} x^{-3/2} \sin[2\varphi(x)] \tag{A-6}$$

$$C(x) = \frac{a^2}{4\pi\sigma}\left(\frac{g}{i}\right)^{1/2} x^{-3/2} \cos[2\varphi(x)] - \frac{a^2 g}{16\pi\sigma^2} x^{-2} \sin[2\varphi(x)]. \tag{A-7}$$

Use of (A-5), (A-6) and (A-7) gives the left side of (A-1) as

$$\begin{aligned}
&\cos 2\sigma t \Big\{ -\frac{3a^2\sigma}{\pi}\left(\frac{g}{i}\right)^{1/2} x^{-3/2}\cos[2\varphi(x)] \\
&\quad + 5/2\, a^2 g/\pi\ x^{-2}\sin[2\varphi(x)] \\
&\quad + 27/40\ \frac{a^2 g}{\pi\sigma}(gi)^{1/2} x^{-5/2}\cos[2\varphi(x)] \\
&+ \sin 2\sigma t \Big\{ \frac{3a^2\sigma}{\pi} x^{-3/2}\sin[2\varphi(x)] \\
&\quad + 5/2\, a^2 g/\pi\, x^{-2}\cos[2\varphi(x)] \\
&\quad - 27/40\ \frac{a^2 g}{\pi\sigma}(gi)^{1/2} x^{-5/2}\sin[2\varphi(x)] \Big\} \\
&+ \Big[a^2\sigma/\pi \left(\frac{g}{i}\right)^{1/2} x^{-3/2}\cos[2\varphi(x)] \\
&\quad - 3/2\, a^2 g/\pi\ x^{-2}\sin[2\varphi(x)] \\
&\quad - \frac{a^2 g}{\pi\sigma}(gi)^{1/2} x^{-5/2}\cos[2\varphi(x)] \\
&\quad + 1/4\ \frac{a^2 g^2}{\pi\sigma^2}\, i x^{-3}\sin[2\varphi(x)] \Big].
\end{aligned} \tag{A-8}$$

Examination of (A-4) and (A-8) shows that use of the expressions for the coefficients A(x), B(x) and C(x) does not yield an exact solution. Differences occur in the third term constants in each of the coefficients of $\cos 2\sigma t$ and $\sin 2\sigma t$. In addition, the segment of (A-8) independent of t contains a fourth term not present in (A-4).

Instinctively, these differences would seem to limit the accuracy of the solution. Iwagaki and Sakai substantiated their solution, however, by considering the relative significance of the terms. It was noted that the terms of the coefficients of sin $2\sigma t$ and cos $2\sigma t$ involved the values $x^{-3/2}$, x^{-2} and $x^{-5/2}$, while the fourth term independent of t in (36) contained x^{-3}. Using $\sigma = 2\pi/T$ and comparing the second, third and fourth terms to the first, the following ratios were found,

$$\text{2nd Term/1st Term} \sim \left[\left\{(gT/2\pi)/(gh)^{1/2}\right\} i\right]^{1},$$

$$\text{3rd Term/1st Term} \sim \left[\left\{(gT/2\pi)/(gh)^{1/2}\right\} i\right]^{2},$$

and

$$\text{4th Term/1st Term} \sim \left[\left\{(gT/2\pi)/(gh)^{1/2}\right\} i\right]^{3}.$$

The interpretation of these ratios is that the successively higher terms become relatively smaller in proportion to i. Iwagaki and Sakai considered the specific case of $i = \frac{1}{10}$, $h = 20$ cm and $T = 3$ sec. Examination of the first and third terms show that they are in phase. For these particular conditions, the ratio between the third and first terms is less than $\frac{1}{10}$. The conclusion is that the difference between the constants for the third terms is negligible.

Similarly, the fourth term is compared to the second, which is in phase with it, and seen to be negligible. The use of (A-5), (A-6) and (A-7) can therefore be used to provide an accurate expression for $\eta^{(2)}$ as

$$\begin{aligned}\eta^{(2)}(x,t) = {} & \cos 2\sigma t \left\{ -\frac{a^2}{\pi i} x^{-1} \sin[2\varphi(x)] - \frac{3}{10}\frac{a^2}{\pi\sigma}\left(\frac{g}{i}\right)^{1/2} x^{-3/2} \cos[2\varphi(x)] \right\} \\ & + \sin 2\sigma t \left\{ -\frac{a^2}{\pi i} x^{-1} \cos[2\varphi(x)] + {}^{3}\!/_{10}\, \frac{a^2}{\pi\sigma} (g/i)^{1/2} x^{-3/2} \sin[2\varphi(x)] \right\} \\ & + \left[{}^{1}\!/_{4}\, \frac{a^2}{\pi\sigma} (g/i)^{1/2} x^{-3/2} \cos[2\varphi(x)] - {}^{1}\!/_{16}\, \frac{a^2 g}{\pi\sigma^2} x^{-2} \sin[2\varphi(x)] \right]. \end{aligned} \qquad \text{(A-9)}$$

The final solution for $\eta^{(2)}(x,t)$ offered by Iwagaki and Sakai was

$$\begin{aligned}\eta^{(2)}(x,t) = {} & \frac{a^2}{\pi i} x^{-1} \cos\left\{ 2\left[\sigma t + 2\sigma\left(\frac{x}{gi}\right)^{1/2} - \pi/4\right] + \pi/2 + \tan^{-1}\left({}^{3}\!/_{10}\, \frac{(gi)^{1/2}}{\sigma} x^{-1/2} \right) \right\} \\ & + \left[\frac{a^2}{4\pi\sigma}\left(\frac{g}{i}\right)^{1/2} x^{-3/2} \cos\left\{ 2\left[2\sigma\left(\frac{x}{gi}\right)^{1/2} - \pi/4\right] \right\} - \frac{a^2 g}{16\pi\sigma^2} x^{-2} \sin\left\{ 2\left[2\sigma\left(\frac{x}{gi}\right)^{1/2} - \frac{\pi}{4}\right] \right\} \right]. \end{aligned} \qquad \text{(A-10)}$$

APPENDIX B

COMPUTER PROGRAMS

The results which are summarized in Table 1 were calculated through the use of a series of simple computer programs. They are essentially a series of do-loops designed to perform a number of iterations over an interval. The results are examined to determine the desired solutions.

The first two programs used are concerned with establishing a value for the quantity, a, found in equations (56), (57) and (63). As was discussed in Section V.E., this evaluation can be made through the use of (67), (69), (71), (72) and (70). The first step in the process is to employ (72),

$$H_1/h_1 = (H_1/H_o)(H_o/L_o)/(h_1/L_o), \tag{72}$$

to find H_1/h_1. In this expression, h_1/L_o is known for the particular case under consideration. Equation (62) gives this term as $h_1/L_o = 0.0157$ for the limiting condition of $h/L \leq 1/20$. Entry into the wave energy flux curves of hyperbolic wave theory with the value of $h_1/L_o = 0.0157$ gives a value of $H_1/H_o = 1.28$. The remaining quantity in the right side of (72), H_o/L_o, is a known value for the specific wave conditions. It can be determined from deep water observations of the wave. Therefore, once this value is specified for a certain set of wave conditions, (72) yields a value of H_1/h_1. For instance, when H_o/L_o is equal to 0.001, H_1/h_1 = (1.28)(.001)/.0157 = 0.0815.

Equation (69) provides a relation for the quantity b as $b = A_1^{(2)}/A_1^{(1)}$. Use of (67), which defines $A_1^{(2)}$ and $A_1^{(1)}$, allows this equation to be evaluated as

$$b = \pi^{-1/4} 2^{1/4} i^{-1/2} (h_1/L_o)^{5/4} a/h_1 . \tag{B-1}$$

Since $\frac{h_1}{L_o}$ = 0.0157 and $L_o = \frac{gT^2}{2\pi}$, (B-1) is equivalent to

$$b = \pi^{3/4} 2^{5/4} i^{-1/2} (h_1/L_o)^{5/4} \frac{a}{gT^2(0.0157)} . \tag{B-2}$$

The next value which must be determined is that of $f(\theta_c)$ in (70) which is defined in (69) as

$$f(\theta_c) = \cos\theta_c - b\sin 2\theta_c .$$

Use of the identity

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$

reduces (69) to

$$f(\theta_c) = \cos\theta_c - \frac{2b\tan\theta_c}{1+\tan^2\theta_c} . \tag{B-3}$$

Equation (71) gives θ_c as

$$\theta_c = \arcsin\left\{\left(\frac{1}{4b} - \left(\frac{1}{16b^2} + 2\right)^{1/2}\right)/2\right\}.$$

Defining x as,

$$x = \left(\frac{1}{4b} - \left(\frac{1}{16b^2} + 2\right)^{1/2}\right)/2 \quad . \tag{B-4}$$

and using the identities which exist when α = arc sin x of

$$\left.\begin{aligned} &\cos\alpha = \sqrt{1-x^2} \\ \text{and} \qquad &\tan\alpha = \frac{x}{\sqrt{1-x^2}}, \end{aligned}\right\} \tag{B-5}$$

results in (B-3) being given by

$$F(\theta_c) = \sqrt{1-x^2} - \frac{2b\,\dfrac{x}{\sqrt{1-x^2}}}{1+\left(\dfrac{x}{\sqrt{1-x^2}}\right)^2} \; . \tag{B-6}$$

$A_1^{(1)}$ is determined from (67) as

$$A_1^{(1)} = 2^{3/4}\pi^{1/4}\, i^{1/2}\left(h_1/L_0\right)^{3/4}\frac{a}{(0.0157)gT^2} \; . \tag{B-7}$$

Equations (B-6) and (B-7) can now be combined to express H_1/h_1 in terms of the quantity a. The requirement then is that of determining the value of a for which the right side of (70) equals the value of H_1/h_1 as given by (72).

The first computer program used defines a function F as the difference of the right hand value of (70), obtained through (B-6) and (B-7), and the left side evaluated by (72). The program is designed to plot F as a function of a. The value of a for which F equals zero is the required solution of (70). Examination of the graph produced by the first program can thus give a first approximation to the quantity a.

The second computer program employed is simply an iteration routine used to refine the value of a. It begins with a first estimate of a determined from the graph of F versus a. The value of a is then incremented in steps of 0.001 and the corresponding values of F calculated. The program is designed to determine the value of F closest to zero and to print the value of a for this case. This is the value of a which satisfied (70).

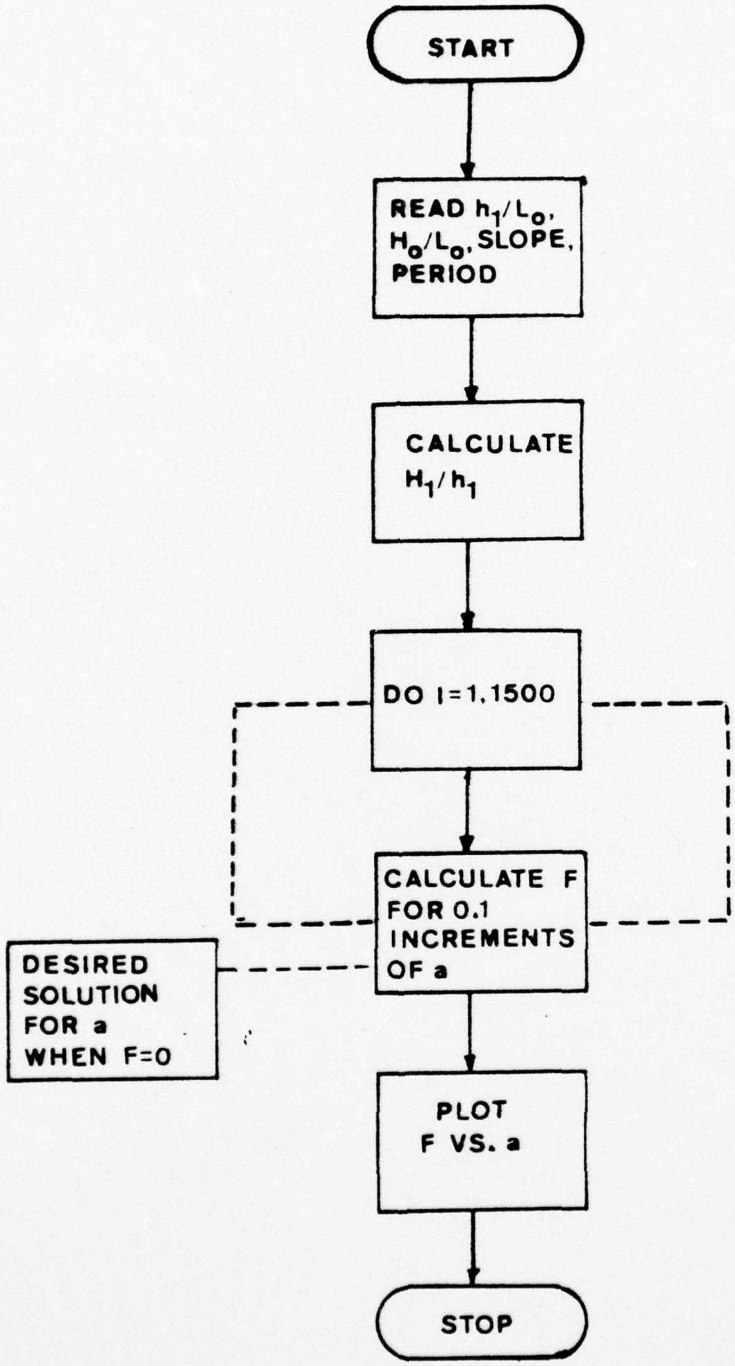
The next series of programs are designed to determine the maximum value of x at which the kinematic breaking criterion is satisfied and the time at which this occurs. This can be accomplished by finding a solution set of (x,t) combinations for which the difference between the right sides of equations (63) and (57) is equal to zero. The technique used is similar to that used in establishing the value of a. The presence of two dependent variables, however, makes the procedure somewhat more complex.

The first program in this determination utilizes the method employed for the plot of F as a function of a. A function is defined which equals the difference between (63) and (57), (c-u). This is plotted as a function of x for a series of times t. Examination of the graphs provides a first guess at the maximum x for which the kinematic breaking condition is satisfied and the time at which it occurs. As was the case for determining a, a second program is now utilized to obtain a more refined solution. Here, time is varied over one second in steps of 0.05 seconds and x is incremented in intervals of 0.01 feet. The value of the function defined as (c-u) is printed for these specific (x,t) combinations. These results are examined to determine the point at which the kinematic criterion is first satisfied.

These first programs have thus determined the maximum x at which the kinematic breaking criterion is satisfied and the time at which it occurs. The remaining programs utilize this result to produce the breaking criteria summarized in Table 1. The fifth program in the complete series simply calculates the value of η (the free surface) and the depth of the bottom for the point specified for the solution of (x_b,t). This is accomplished by using (56) to define η and $h = i \cdot x_b$. The ratios of η_b/h_b and

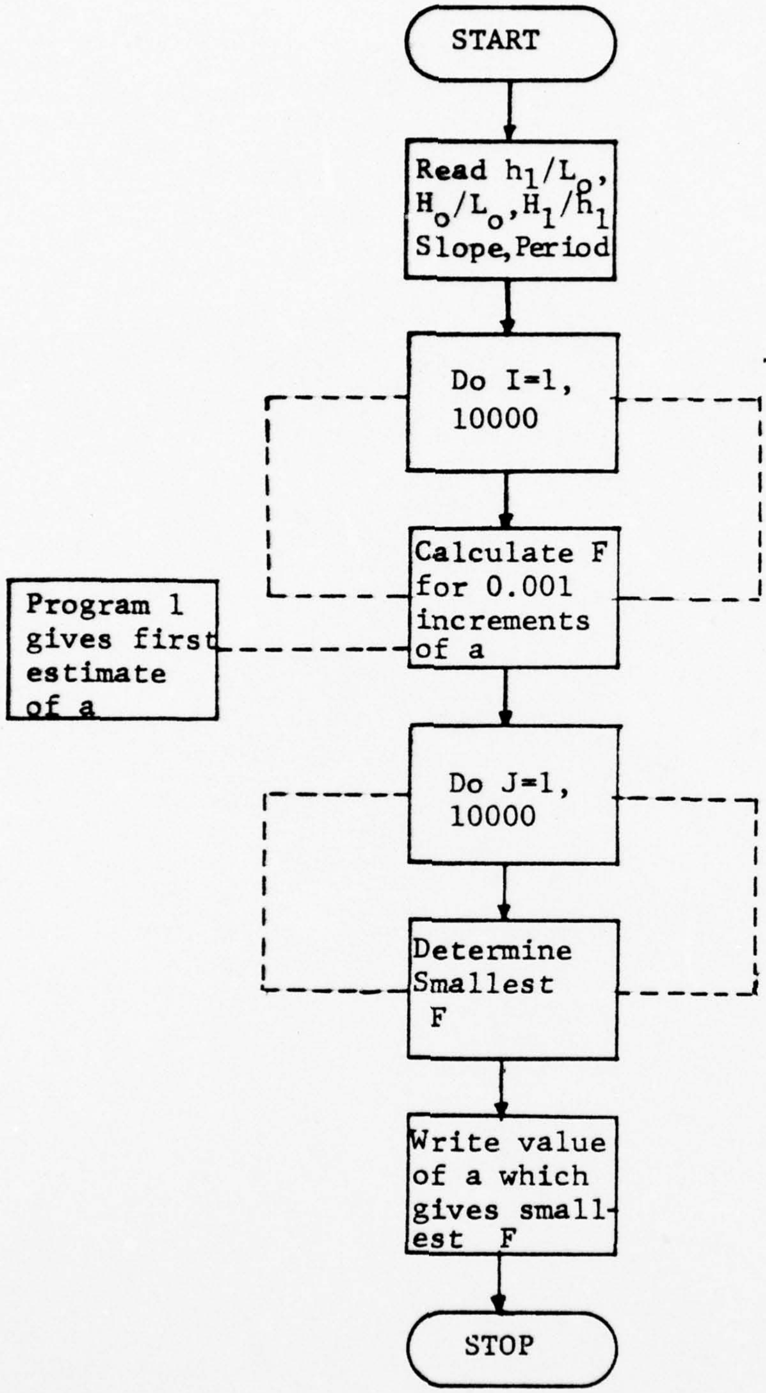
and h_b/L_o are then calculated. A similar program is also provided which graphs η as a function of x.

Pages 86-91 contain flow charts for the programs employed in these determinations.



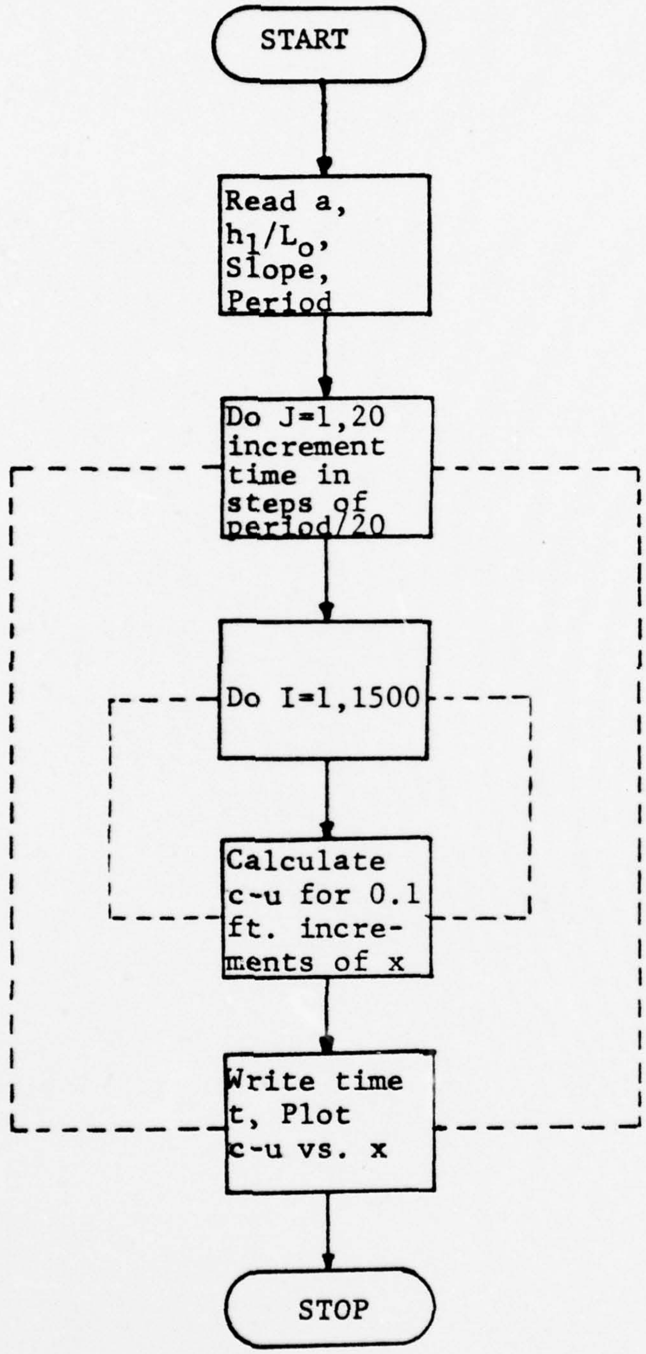
PROGRAM 1: APPROXIMATION OF a

Figure 9



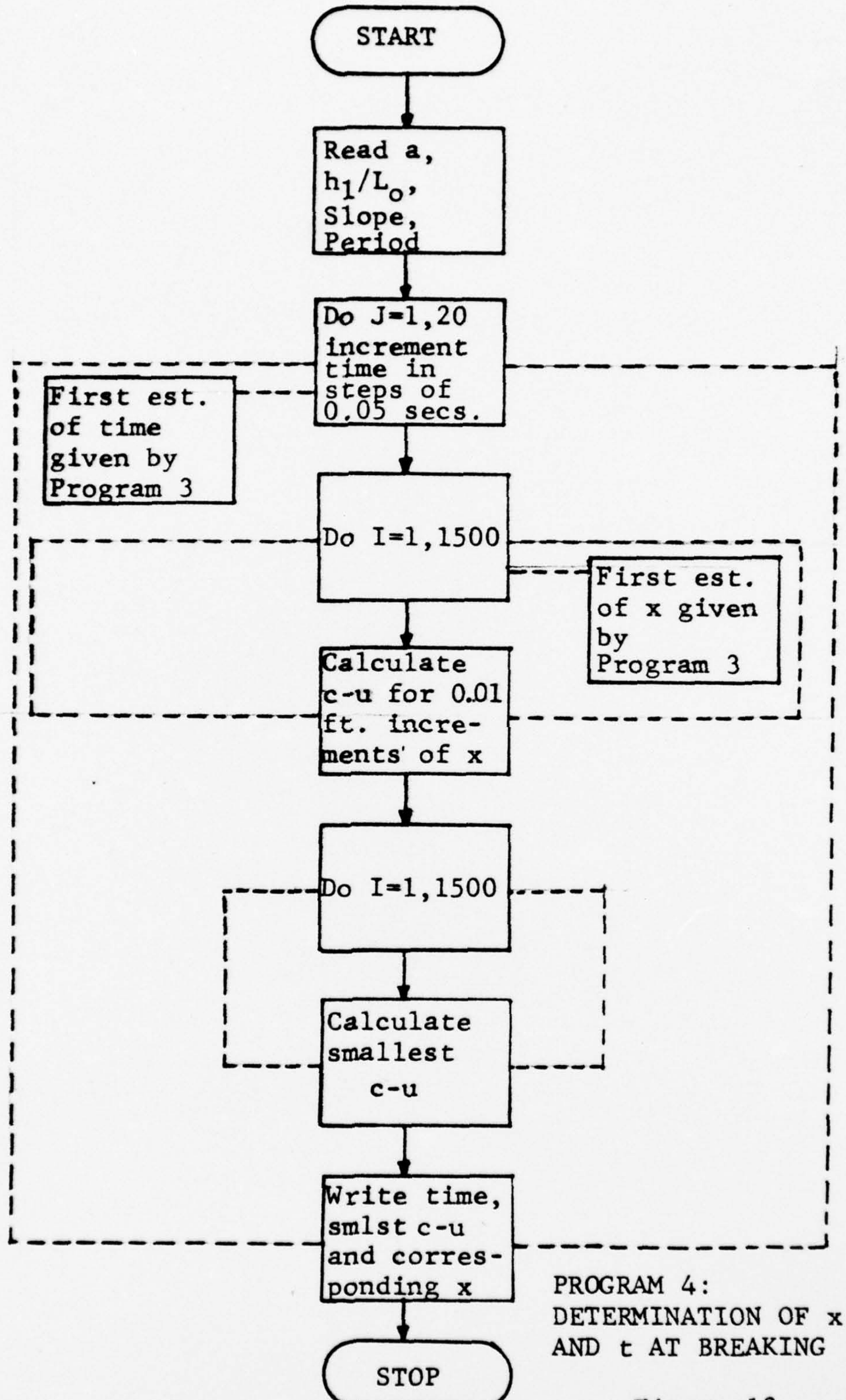
PROGRAM 2: REFINEMENT OF a

Figure 10



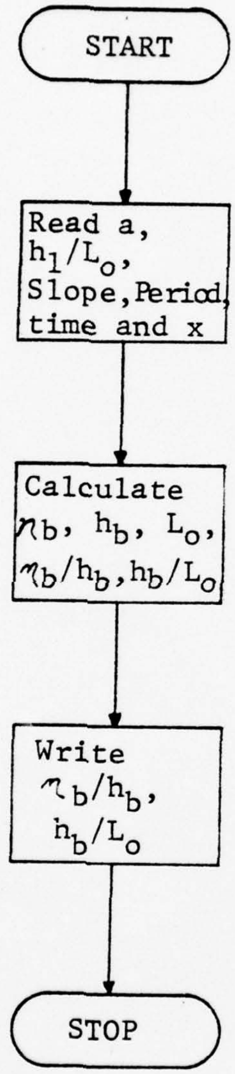
PROGRAM 3: FIRST APPROXIMATION OF x AND t AT BREAKING

Figure 11



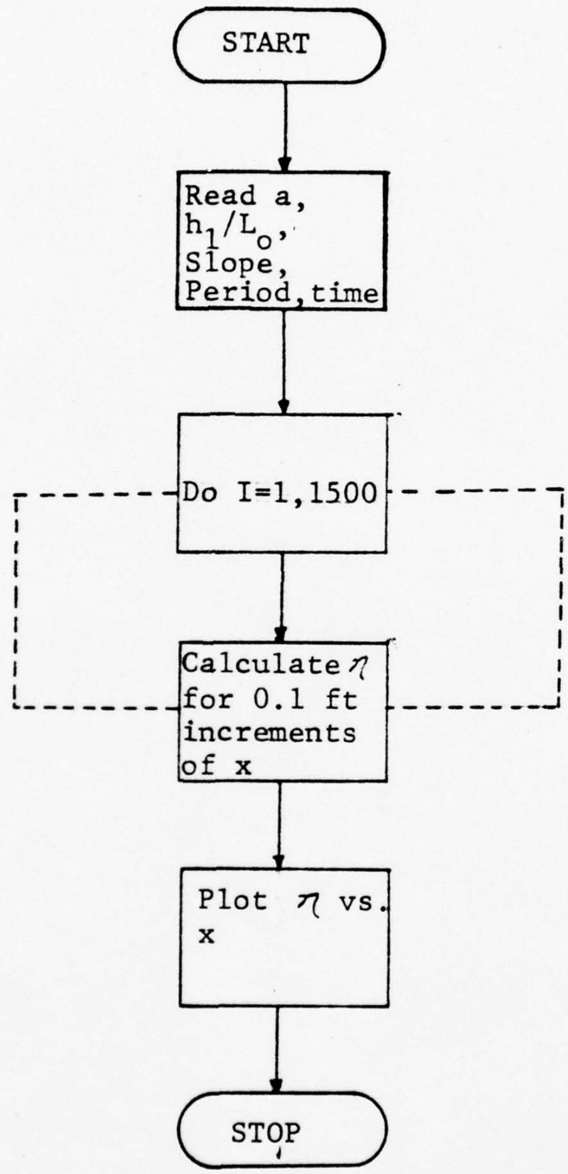
PROGRAM 4:
DETERMINATION OF x
AND t AT BREAKING

Figure 12



PROGRAM 5: CALCULATION OF η_b/h_b AND h_b/L_o

Figure 13



PROGRAM 6: PLOT OF η AT TIME OF BREAKING

Figure 14

LIST OF REFERENCES

Ayyar, H. R., "Periodic Waves Shoaling in Waters over Steeply Sloping Bottoms," in Proceedings of the Twelfth Coastal Engineering Conference, v. 1, p. 363-369, 1970.

Benney, D. J., "Long Non-Linear Waves in Fluid Flows," Journal of Mathematics and Physics, v. 45, p. 52-63, 1966.

Boussinesq, J., "Theorie de l'intumescence liquide appelee onde solitaire ou de translation se propageant dans un canal rectangulaire," Institut de France, Academie des Sciences, Comptes Rendus, p. 755, June 19, 1871.

Boussinesq, J., "Theories des ondes et de remous qui se porpagent le long d'un canal rectangulaire horizontal, en communiquant ou liquide contenu dans ce canal des vitesses sensiblement paralleles de la surface au fond," Journal Mathematical Pures Appliquess, v. 17, p. 55-108, 1872.

Burger, W., "A Note on the Breaking of Waves on Non-Uniformly Sloping Beaches," Journal of Mathematics and Mechanics, v. 16, p. 1131-1142, 1967.

Carrier, G. G. and Greenspan, H. P., "Water Waves of Finite Amplitude on a Sloping Beach," Journal of Fluid Mechanics, v. 4, p. 97-109, 1958.

Chappelear, J. E., "On the Theory of the Highest Waves," Beach Erosion Board Technical Memo No. 116, July, 1959.

Davies, T. V., "The Theory of Symmetrical Gravity Waves of Finite Amplitude," in U. S. National Bureau of Standards Circular No. 521, p. 55-60, 1952.

De, S. C., "Contributions to the Theory of Stokes Waves," Proceedings on the Cambridge Philosophical Society, v. 51, p. 713-736, 1955.

Dean, R. G. and Eagleson, P. S., "Finite Amplitude Waves," in Estuary and Coastline Hydrodynamics, Ippen, A. T., editor, McGraw-Hill, 1966.

Dean, R. G., "Breaking Wave Criteria: A Study Employing Numerical Wave Theory," in Proceedings of the Eleventh Conference on Coastal Engineering, v. 1, p. 108-123, 1968.

Freidrichs, K. O., "On the Derivation of the Shallow Water Theory," Appendix to "The Formation of Breaker and Bores," Stoker, J. J., Communications on Pure and Applied Mathematics, v. 1, p. 81-85, 1948.

Gaughan, M. K., Komar, P. D., and Nath, J. H., Breaking Waves: A Review of Theory and Measurements, Oregon State University, 1973.

Goda, Y., "A Synthesis of Breaker Indices," Transactions of the Japanese Society of Civil Engineers, v. 2, p. 227-230, 1970.

Greenspan, H. P., "On the Breaking of Water Waves of Finite Amplitude on a Sloping Beach," Journal of Fluid Mechanics, v. 4, p. 330-334, 1958.

Gwythery, R. F., "The Classes of Long Progressive Waves," Philosophical Magazine, v. 50, p. 213-216, 1900.

Havelock, E. T., "Periodic Irrotational Waves of Finite Height," Proceedings of the Royal Society, v. 95, p. 38-51, 1918.

Iwagaki, Y. and Sakai, T., "Shoaling of Finite Amplitude Long Waves on a Beach of Constant Slope," in Proceedings of the Thirteenth Coastal Engineering Conference, v. 1, p. 347-364, July, 1972.

Keller, J. B., "The Solitary Wave and Periodic Waves in Shallow Water," Communications on Pure and Applied Mathematics, v. 1, p. 323-339, 1948.

Kinsman, B., Wind Waves, Prentice-Hall, 1965.

Korteweg, D. J. and de Vries, G., "On the Change of Form of Long Waves Advancing in a Rectangular Canal, and on a New Type of Long Stationary Waves," Philosophical Magazine, v. 5, p. 442-443, 1895.

Laitone, E. V., "The Second Approximation to Cnoidal and Solitary Waves," Journal of Fluid Mechanics, v. 9, p. 430-444, 1960.

Laitone, E. V., "Higher Approximations to Non-Linear Water Waves and the Limiting Heights of Cnoidal, Solitary and Stokes' Waves," Beach Erosion Board Technical Manual No. 133, February, 1963.

LéMehauté, B., "Mass Transport in Cnoidal Waves," Journal of Geophysical Research, v. 73, p. 5973-5979, 15 September 1968.

LéMehauté, B., "Water Wave Theories," Environmental Science Service Administration Technical Report ERL 118-POL-3-2 An Introduction to Hydrodynamics and Water Waves Vol. II, July, 1969.

Lenau, C. W., "The Solitary Wave of Maximum Amplitude," Journal of Fluid Mechanics, v. 26, p. 309-320, 1966.

McCowan, J., "On the Highest Wave of Permanent Type," Philosophical Magazine, v. 38, p. 351-357, 1894.

Mei, C. D., "On the Propagation of Periodic Water Waves Over Beaches of Small Slope," Coastal Engineering Research Center Technical Note No. 12, October, 1966.

Michell, J. H., "On the Highest Waves in Water," Philosophical Magazine, v. 38, p. 351-357, 1893.

Packham, B. A., "The Theory of Symmetrical Gravity Waves of Finite Amplitude, Part 2 The Solitary Wave," Proceedings of the Royal Society, v. 213, p. 238-249, 1952.

Peregrine, D. H., "Equations for Water Waves and the Approximations Behind Them," in Waves on Beaches and Resulting Sediment Transport, Meyer, R. Ed., editor, p. 95-122, Academic Press, 1972.

Rankine, W. J., "Summary of Properties of Certain Streamlines," Philosophical Magazine, v. 28, p. 282-288, 1894.

Rayleigh, Lord, "On Waves," Philosophical Magazine, v. 1, p. 257-279, 1876.

Riabouchinsky, D., "Sur l'Analogie Hydraulique des Mouvements d'un Fluide Compressible," Institut de France, Academie des Sciences-Comptes Rendus, v. 195, p. 998, 1932.

Stoker, J. J., "The Formation of Breakers and Bores," Communications on Applied Mathematics, v. 1, p. 1-87, 1948.

Stoker, J. J., Water Waves, Interscience Publishers, 1957.

Stokes, G. G., "On the Theory of Oscillatory Waves," Transactions of the Cambridge Philosophical Society, v. 8, p. 441-455, 1 March 1847.

Stokes, G. G., Mathematical and Physical Papers, Cambridge Univeristy Press, 1880.

Tlapa, G., Mei, C. C., and Eagleson, P. S., "The First Order Theory for Infinitesimal Waves on a Mildly Sloping Beach," Appendix A to "On the Propagation of Periodic Water Waves Over Beaches of Small Slope, MIT Hydrodynamics Laboratory Technical Note No. 12, October 1966.

Ursell, F., "The Long-Wave Paradox in the Theory of Gravity Waves," Proceedings of the Cambridge Philosophical Society, v. 39, p. 685-694, 1953.

Yamada, J., "On the Highest Solitary Wave," Report of Research Institute for Applied Mechanics, v. 5, p. 53-155, 1957.

Yamada, H., Kimura, G., and Okabe, J., "Precise Determination of the Solitary Wave of Extreme Height on Waters of a Uniform Depth," Report of Research for Applied Mechanics, v. 16, p. 15-32, 1968.

AD-A032 323 NAVAL POSTGRADUATE SCHOOL MONTEREY CALIF F/G 20/14

BREAKING WAVE CRITERION ON A SLOPING BEACH.(U)

SEP 76 R M SMITH

UNCLASSIFIED NL

2 OF 2
ADA032323

END

DATE
FILMED
1 - 77

INITIAL DISTRIBUTION LIST

		No. Copies
1.	Defense Documentation Center Cameron Station Alexandria, Virginia 22314	2
2.	Library, (Code 0212) Naval Postgraduate School Monterey, California 93940	2
3.	Department Chairman, Code 58 Department of Oceanography Naval Postgraduate School Monterey, California 93940	3
4.	Assoc. Professor E. B. Thornton, Code 58 Department of Oceanography Naval Postgraduate School Monterey, California 93940	5
5.	Assoc. Professor J. J. von Schwind, Code 58 Department of Oceanography Naval Postgraduate School Monterey, California 93940	1
6.	Oceanographer of the Navy Hoffman Building No. 2 200 Stovall Street Alexandria, Virginia 22332	1
7.	Office of Naval Research Code 462 Arlington, Virginia 22217	1
8.	Dr. Robert E. Stevenson Scientific Liaison Office, ONR Scripps Institution of Oceanography La Jolla, California 92037	1
9.	Library, Code 3330 Naval Oceanographic Office Washington, D. C. 20373	1

10. SIO Library 1
University of California, San Diego
P. O. Box 2367
La Jolla, California 92037

11. Department of Oceanography Library 1
University of Washington
Seattle, Washington 98105

12. Department of Oceanography Library 1
Oregon State University
Corvallis, Oregon 97331

13. Commanding Officer 1
Fleet Numerical Weather Central
Monterey, California 93940

14. Commanding Officer 1
Environmental Prediction Research Facility
Monterey, California 93940

15. Department of the Navy 1
Commander Oceanographic System Pacific
Box 1390
FPO San Francisco 96610

16. LT Richard M. Smith 3
8245 Van Noord Avenue
North Hollywood, California 91605

17. Commander, Naval Weather Service Command 1
Naval Weather Service Headquarters
Washington Navy Yard
Washington, D. C. 20390

18. Coastal Studies Institute 1
Louisiana State University
Baton Rouge, Louisiana 70803

19. School of Marine and Atmospheric Science 1
University of Miami
Miami, Florida 33149

20. LT V. S. Hulstrand 1
SMC Box 2489
Naval Postgraduate School
Monterey, California 93940